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Derivation of the Elementary Particles Total Energy by Quantum Statistical Physics and Generalized Special Relativity

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ABSTRACT: Einstein theory of relativity is one of big achievements in physics. This requires using it for promoting physical theories. Special relativistic energy momentum equation was utilized to determine the Fermi energy. The results obtained indicate that the Fermi energy is rest mass dependent. Unfortunately special relativity suffers from the lack of a term which feels the existence fields. This defect was cured by constructing potential dependent special relativity. This potential dependent special relativity was used to find the self energy of gravitational mass under equilibrium conditions and to find also the Fermi energy. It was found that Fermi self energy is dependent on the rest mass and mass radius.

KEYWORDS: Generalized special relativity, Fermi potential, Fermi energy, rest mass, atomic radius, sphere of radius.

INTRODUCTION:

we investigated the statistical thermodynamics of ideal gas, we employed quantum statistical to deal with the non-translational degrees of freedom let us now discuss ideal gases from a purely quantum standpoint it turns out that this approach is necessary in order to treat either very low-temperature density. Furthermore, it also allows us to investigate completely non-classical gases such as photons or the conduction electron in a metal [1,2,3,4,5]. We employed classical mechanics to deal with the translational degrees of freedom of the constituent particles and Generalized Special Relativity to deal with the non-translational degrees of freedom. Let us now discuss ideal gases from a purely generalized special relativity and quantum statistical standpoint. It turns out that this approach is necessary to deal with either low temperature or high density gases.

Quantum Statistical Physics:

The combination of position space and momentum space is called phase-space [6]. Thus a phase space is six dimensional space, three position coordinates and three momentum coordinates: $p = p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k}$ (1)

Uncertainty principle:

$$\Delta x\Delta p_x \backsim \Delta y\Delta p_y \backsim \Delta z\Delta p_z \backsim h$$

The minimum size of phase call volume in given according as uncertainty principle as:

$$\tau = (\Delta x\Delta y\Delta z)(\Delta p_x\Delta p_y\Delta p_z) \backsim h^3 \quad (2)$$

For a single particle we have six dimensional phase-spaces, hence the total volume in phase space is given as:

$$V_{ph} = \int dxdydz \int dp_{F_x}\,dp_{F_y}dp_{F_z}$$

Where limit of integration are to be specified

$$\int dxdydz = V$$

Phase space volume:

$$V_{ph} = V\int dp_{F_x}\,dp_{F_y}dp_{F_z} \quad (3)$$

Hence, the number of cells phases space:

$$\tau = \frac{phase\ space\ \ volume}{phase\ cell\ volume} = \frac{N}{2}$$

$$\frac{V\int dp_{F_x}\,dp_{F_y}dp_{F_z}}{h^3} = \frac{N}{2} \quad (4)$$

Where: N The total number of particles.

Let us assume that the ideal gas is highly degenerate, which is equivalent to taking the limit $T \to 0$. In this case, we know, from the previous section, that the Fermi momentum can be written [3]:

$$\int dp_{F_x}\,dp_{F_y}dp_{F_z} = \frac{h^3}{2}\left(\frac{N}{V}\right) \quad (5)$$

The volume of the sphere of radius p_F in p- space is:

$$p = \frac{4}{3}\pi p_F^3 \quad (6)$$

Hence

$$\frac{4}{3}\pi p_F^3 = \frac{h^3}{2}\left(\frac{N}{V}\right) \quad (7)$$

Where

$$p_F^3 = \frac{3h^3}{8\pi}\left(\frac{N}{V}\right) = \frac{3\pi^2 h^3}{8\pi^3}\left(\frac{N}{V}\right) = 3\pi^2\hbar^3\left(\frac{N}{V}\right) \quad (8)$$

Thus

$$p_F = (3\pi^2)^{1/3}\hbar\left(\frac{N}{V}\right)^{1/3} \quad (9)$$

The Fermi momentum can be written:

$$p_F = \Lambda\left(\frac{N}{V}\right)^{1/3} \quad (10)$$

Where

$$\Lambda = (3\pi^2)^{1/3}\hbar \quad (11)$$

But for spherical body:

$$V = \frac{4}{3}\pi R^3 \quad (12)$$

Furthermore, the number of particles states contained in an annular radius of p-space lying between radii p and $p + dp$ is [3]:

$$\int dp_{F_x}\,dp_{F_y}dp_{F_z} = 4\pi p_F^2 dp$$

There: $dN = 2 \times$ the number of phase cells:

$$dN = \frac{2V\int dp_{F_x}\,dp_{F_y}dp_{F_z}}{h^3} = \frac{2V(4\pi p_F^2 dp)}{h^3}$$

The number of quantum states in a phase space, from equation:

$$dN = \frac{8\pi V p^2}{h^3}dp \quad (13)$$

Where:

$$p = \frac{h}{\lambda} = \frac{h\nu}{c} \quad , \quad dp = \frac{hd\nu}{c} \quad (14)$$

From equations (13-14), the number modes per unit volume within the frequency range ν and $\nu + d\nu$ is:

$$dN = \frac{8\pi V\nu^2}{c^3}d\nu \quad (15)$$

Where:

$$\omega = 2\pi\nu \quad , \quad d\omega = 2\pi d\nu$$

From equation (15) we find the angular frequency is:

$$dN = \frac{8\pi V}{c^3}\left(\frac{\omega}{2\pi}\right)^2\frac{d\omega}{2\pi}$$

$$dN = \frac{V}{\pi^2 c^3}\omega^2 d\omega$$

$$\int_0^N dN = \frac{V}{\pi^2 c^3}\int_0^\omega \omega^2\,d\omega$$

$$N = \frac{V}{3\pi^2 c^3}\omega^3 \quad (16)$$

Thus:

$$\omega_F = c\left(3\pi^2\frac{N}{V}\right)^{1/3} \quad (17)$$

But:

$$\omega_F = ck_F \quad (18)$$

That is:

$$k_F = \left(3\pi^2\frac{N}{V}\right)^{1/3} \quad (19)$$

Thus:

$$p_F = \hbar k_F = (3\pi^2)^{1/3}\hbar\left(\frac{N}{V}\right)^{1/3} \quad (20)$$

Thus at: $T = 0$ all quantum states with $k < k_F$ are filled, and all those with $k > k_F$ are empty. The volume of the sphere of radius k_F in k-space is:

$$k = \frac{4}{3}\pi k_F^3 \quad (21)$$

Quantum mechanics presents us with a relationship between frequency f of radiation and the energy of each photon:

$$E_F = \hbar\omega_F \quad (22)$$

According to equations (11), (17) and (22) the Fermi energy at: $= 0$, takes the form

$$E_F = \Lambda c\left(\frac{N}{V}\right)^{1/3} \quad (23)$$

According to the law of quantum mechanics for particle in box the energy is given by:

$$E_F = c_0 V^{-1/3} \quad (24)$$

Where:

$$c_0 = \Lambda c N^{1/3} \quad (25)$$

In Fermi momentum space:

$$p_F = p_{F_x} + p_{F_y} + p_{F_z} \quad (26)$$

The energy corresponding to momentum p_F in E_F is:

$$p_F = (2mE_F)^{1/2} \quad (27)$$

So volume of that space is:

$$\int dp_{F_x}\,dp_{F_y}dp_{F_z} = \frac{4}{3}\pi p_F^3 = \frac{4}{3}\pi(\,2mE_F)^{3/2} \quad (28)$$

Hence the number of cells in the phase space

$$\tau = \frac{4\pi}{3h^3}V(2mE_F)^{3/2} \quad (29)$$

For a single particle the number of accessible microstate will be equal to the number of cells in the phase space. Therefore the number of microstate in the energy range E and $(E + dE)$ is:

$$\frac{d\tau}{dE} = \frac{2\pi}{h^3} V(2m)^{3/2} E^{1/2} \qquad (30)$$

It is easily demonstrated that $E_F \gg k_B T$ for conventional metals at room temperature. Hence the total kinetic energy, the kinetic energy cell gas can be written:

$$K_E = \int \frac{p_F^2}{2m} dN \qquad (31)$$

From equation (14):

$$K_E = \frac{3V}{\Lambda^3} \int_0^{p_F} \frac{p_F^2}{2m} p_F^2 dp = \frac{3}{5} \frac{V}{\Lambda^3} \frac{p_F^5}{2m} \quad (32)$$

Substituting in equation (10) we get:

$$K_E = \frac{3}{5} \frac{V\Lambda^2}{2m} \left(\frac{N}{V}\right)^{5/3} = \frac{3}{5}\left[\frac{\Lambda^2}{2\rho}\left(\frac{N}{V}\right)^{5/3}\right] \quad (33)$$

The relativistic between the total energy E, momentum p and rest mass m_0 :

$$E = (p^2c^2 + m_0^2c^4)^{1/2} \qquad (34)$$

Which for:

$$pc \ll m_0c^2$$

We can also write the total particle energy E using special relativity formula:

$$K_E = \int E\, dN \qquad (35)$$

Using special relativity formula (34) and relations (14) and (35) as follows:

$$K_E = \frac{3V}{\Lambda^3} \int_0^{p_F} (p^2c^2 + m_0^2c^4)^{1/2}\, p_F^2 dp \qquad (36)$$

When: $pc \ll m_0c^2$, to simplify energy relativistic formula to get:

$$E = pc\left(1 + \frac{m_0^2c^2}{p^2}\right)^{1/2} = pc\left(1 + \frac{m_0^2c^2}{2p^2}\right) \quad (37)$$

Thus, particle energy as follows, by analogy with equation (37) thus:

$$K_E = \frac{3Vc}{\Lambda^3} \int_0^{p_F} \left(p^3 + \frac{m_0^2c^2}{2} p + \cdots\right) dp \quad (38)$$

$$K_E = \frac{3Vc}{4\Lambda^3} (p_F^4 + m_0^2c^2p_F^2 + \cdots) \quad (39)$$

Using the tow relation (10) and (12), the Fermi momentum can be written:

$$p_F = \Lambda \left(\frac{3N}{4\pi}\right)^{1/3} \frac{1}{R} \qquad (40)$$

And by reference to the equations (12), (39) and (40) it is clear that:

$$K_E = \frac{3c}{4\Lambda^3}\left(\frac{4\pi R^3}{3}\right)\left(\Lambda^4\left(\frac{3N}{4\pi}\right)^{4/3}\frac{1}{R^4} + m_0^2c^2\Lambda^2\left(\frac{3N}{4\pi}\right)^{2/3}\frac{1}{R^2} + \cdots\right) \quad (41)$$

Generalized Special Relativity:

Consider first the generalized special relativity energy E equilibrium condition by minimizing E with respect to radius [7]:

$$E = \frac{g_{00}m_0c^2}{\sqrt{g_{00} - \frac{v^2}{c^2}}} \qquad (42)$$

Newtonian gravity potential:

$$\varphi = \frac{GM}{R}$$

Thus:

$$g_{00} = 1 - \frac{2\varphi}{c^2} = 1 - \frac{2GM}{c^2R}$$

$$E = \frac{m_0c^2\left(1 - \frac{2GM}{c^2R}\right)}{\sqrt{1 - \frac{2GM}{c^2R} - \frac{v^2}{c^2}}} \qquad (43)$$

Thus:

$$m_0 = \mathrm{M}$$

$$\frac{v^2}{c^2} = \frac{M^2v^2}{M^2c^2} = \frac{p_F^2}{M^2c^2}$$

$$E_F = Mc^2\left(1 - \frac{2GM}{c^2R}\right)\left(1 - \frac{2GM}{c^2R} - \frac{p_F^2}{M^2c^2}\right)^{-1/2} \quad (44)$$

The equilibrium takes place when R is non-negative

$$p_F > Mc$$

For small the gravitational potential and Fermi momentum p_F compared to speed of light i.e:

$$\frac{2GM}{c^2R} < 1 \quad , \quad \frac{p_F^2}{M^2c^2} < 1$$

Thus equation (44) reduced to:

$$E_F = Mc^2\left(1-\frac{2GM}{c^2R}\right)\left(1+\frac{GM}{c^2R}+\frac{p_F^2}{2c^2M^2}\right)$$

$$E_F = Mc^2\left(1+\frac{GM}{c^2R}+\frac{p_F^2}{2M^2c^2}-\frac{2GM}{c^2R}-\frac{2G^2M^2}{c^4R^2}-\frac{Gp_F^2}{Mc^4R}\right)$$

$$E_F = Mc^2\left(1+\frac{p_F^2}{2M^2c^2}-\frac{GM}{c^2R}-\frac{2G^2M^2}{c^4R^2}-\frac{Gp_F^2}{Mc^4R}\right)$$

In the case of weak gravitational field where the term $\left(\frac{G^2M^2}{c^4R^2}\right)$ and $\left(\frac{Gp_F^2}{Mc^4R}\right)$ is neglected:

$$E_F = Mc^2+\frac{p_F^2}{2M}-\frac{GM^2}{R} \qquad (45)$$

The total Fermi energy:

$$E_F = Mc^2+K_E-U \qquad (46)$$

Where: K_E: the kinetic energy, U : the gravitational potential energy.

Therefore, with the aid of equations (33) in equation (46) reads:

$$E_F = Mc^2+\frac{3}{5}\frac{\Lambda^2}{2\rho}\left(\frac{N}{V}\right)^{5/3}-\frac{GM^2}{R} \qquad (47)$$

Let us assume, for the sake of simplicity, that the density of the particle is uniform. In this case, the gravitational potential energy U takes the form:

$$U=-\int_0^R 4\pi r^2\rho\frac{Gm(r)}{r}\,dr$$

Where:

$$m(r)=\frac{4}{3}\pi r^3\rho$$

Hence:

$$U=-\frac{16}{3}\pi^2\rho^2G\int_0^R r^4\,dr$$

$$U=-\frac{16}{15}\pi^2\rho^2GR^5$$

There:

$$\rho=\frac{3m}{4\pi R^3}$$

Thus: $m=M$, $r=R$

Gravitation potential energy:

$$U=-\frac{3}{5}\frac{GM^2}{R} \qquad (48)$$

Therefore equation (48) comparing the relation (47) we find the total energy of elementary particle:

$$E_F = Mc^2+\frac{3}{5}\left[\frac{\Lambda^2}{2\rho}\left(\frac{N}{V}\right)^{5/3}\right]-\frac{3}{5}\left[\frac{GM^2}{R}\right] \qquad (49)$$

($\frac{3}{5}$: uniform density sphere)

Discussion:

Equations (1 - 10) were devoted to find the general expressions for the Fermi energy so as to incorporate the special relativity energy and potential dependent special relativity. Special relativity expression for the total energy for a large number of particles was derived in equation (36). Using special relativity energy expression, the total energy was found to be dependent on the rest mass as well as the atomic radius as shown in equation (41). Equation (40) indicated that the Fermi momentum is dependent on the mass radius of the system. Using energy expressions of the potential dependent special relativity in equation (43), the self-energy at equilibrium was found and the corresponding Fermi energy was found also. The Fermi energy was found to be dependent on the rest mass as well as the atomic radius beside the Fermi potential.

CONCLUSION

Using special relativity energy momentum relation a useful expressions for the Fermi energy was found. The Fermi energy was found to be dependent on the rest mass. The potential dependent special relativity energy momentum relation was used for the Fermi self-energy at equilibrium. The Fermi energy was found to be dependent on the rest mass as well as on the radius.

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