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Modelling and Simulation of Deformation Field Variables in a Concrete Beam Under Pressure Loading

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ABSTRACT: Deformation is determined analytically by the field variables (displacement, stress and strain). Here, we determine the displacement state of a concrete beam using a finite element solver (Abaqus). Concrete beam, a solid mixture of sand, cement, and water is model with its material properties of Young Modulus of 30Mpa, Poisson ratio of 0.2 and pressure load of magnitude 250KN in Abaqus CAE. The results obtained from the simulation of the Force-Displacement relationship are 85KN, 160KN, and 220KN corresponding to the displacement of 2mm, 1.5mm, and 1.0mm respectively. At a force of 85KN, the material extended by 2mm, attaining its elastic limit. The simulation of stress-strain relationship shows that, at a peak force intensity of 52Mpa, the beam undergo a deformation rate of 0.003. Comparing the results obtained from simulation with the analytic result obtained in the form of stiffness matrices, resulting from the finite element analysis of the beam. The stable equilibrium configuration of the beam in the stiffness matrix has all its eigen-values as 0.3, corresponding to the range of the pressure force of 1KN to 85KN and a displacement of 2mm of the simulation. While, higher displacement and deformation occurs when the pressure load is beyond 85KN, which is captured in the eigen-values of stiffness matrix as 0.15, and 0.00 entries. Hence, from the results, the simulated model from the Abaqus is in an excellent agreement in interpreting the state of the beam under pressure load.

KEYWORDS: Deformation, Force, Displacement, Stress, Strain

1. INTRODUCTION

Solid mechanics modelling and simulation has vase applications in industrial design and manufacturing. This is because the development of numerical techniques has increased the computational power which has enabled engineers to study mechanical patterns by creating 3-dimensional models and simulating the behavior of solid materials. Material such properties such stress, strain, and deformation can be simulated and analyzed to maximum level of accuracy.

(*Usik Lee*; 1994)[9] in their findins established an equal solid plate which involved the use of a properly defined finite element matrices for easy analyses of strain and kinetic energies contained in a lattice cell.

Also,

(*Allsandro D.C*, *Antonio B*, *Fransco D*,*and Ivan*; 2017)[2] revealed in their analyses a simple discrete system in order to model structural failure and fracture within the same numerical and theoritical framework. Still under the concept of deformation(*Ismail A.M*; 2024)[6] provides an comprehensive approach of modelling and simulation of complex systems, including queuing and life support systems. To visualise a detailed picture of any system, models are created to mimic the actual one. Conseqeuntly, simulation, enables us to create a real life situations for detailed analyses and effective decision making. While (*Amit Dutt*, *Rub Pant*, *Chandra Mohan*, *and Akkireddy*; 2025)[3] investigated the changing phases between modelling, simulation, and differential systems in the study of continuum mechanics, with an emphasis on the use of numerical methods and the combination of several differential equations. Also, (*Guang Su*, *and Aimin Zhang*; 2021)[5] established the temperature variation model of heat-controlled materials, performed the numerical simulation of the relationship between mechanical properties and heat of temperature-controlled materials, and analyzed the stress-strain relationship. The last simulation experiment showed that temperature-controlled materials have greater chances of greater energy storage density. From the available literature, modelling and simulation of deformation field variable in concrete beam has not been elaborately discussed. Here, in this paper we address this concept with a detailed mathematical analysis in comparism with the simulated result.

2. MATHEMATICAL FORMULATION

The theory of elasticity deals with deformation of solid under the influence of forces. A solid deforms when there is a variation in its displacement field. When this variation is minimal, it will attain a stable equilibrium configuration.

When maximum, the atomic and sub-atomic particles are stressed and strained beyond their elastic limit. Leading to dislocation and the subsequent deformation of the solid.

Displacement being the primary field variable of deformation can be determine by considering the total potential of the solid through calculus of variation as:

$$I\,[u(x)] = \int_a^b F(u, u', x)dx \quad (1)$$

Where I, is the total potential of the solid, u, is the displacement, and x, is the position of the particles within the solid. While the right hand term in equation (1) with definite integral represent the displacement field of the solid.

2.1 Mathematical Analysis of Displacement Field

The displacement field of the solid called functional, is described with a simple diagram below:

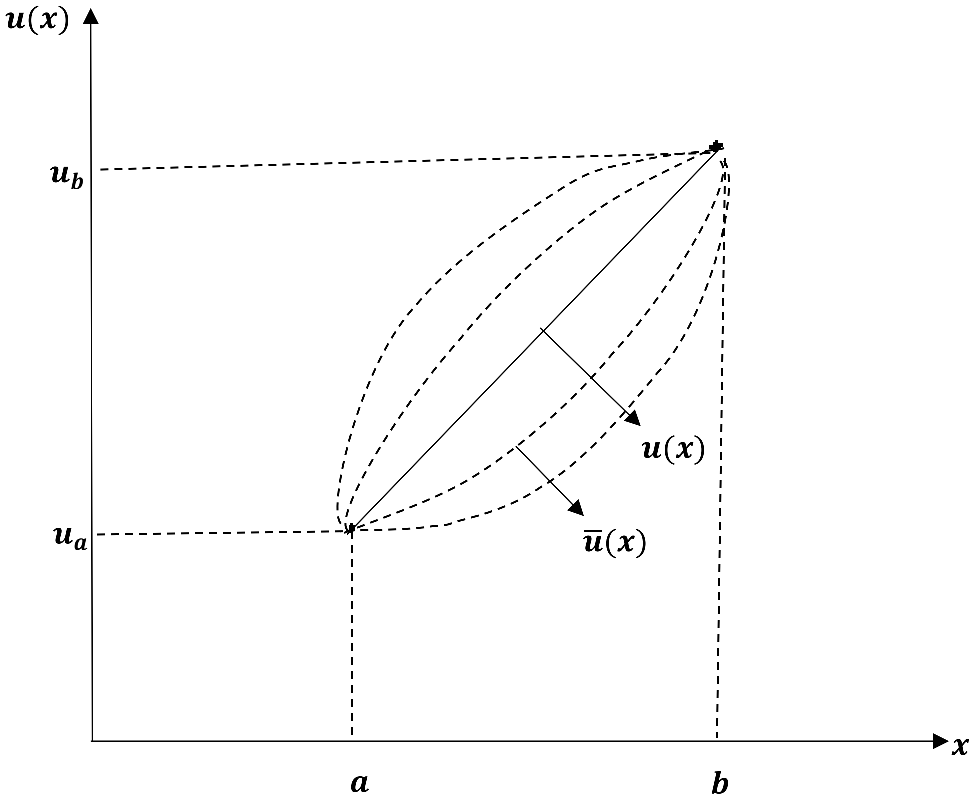

Figure 1: Displacement Description of a Functional

Hence,

$\delta u = \bar{u} - u$

$\bar{u} = u + \delta u \quad (2)$

and

$\delta u' = \bar{u}' - u'$

$\bar{u}' = u' + \delta u' \quad (3)$

Also

$$\delta u' = \delta\left(\frac{du}{dx}\right) = \frac{d}{dx}\delta u \quad (4)$$

Then, the variation of the functional is given as:

$\delta F = F(\bar{u}, \overline{u'}, x) - F(u, u', x) \quad (5a)$

$F(\bar{u}, \overline{u'}, x) = F(u, u', x) + \delta F \quad (5b)$

Replacing $\bar{u}$ *and* $\overline{u'}$ in equation (5b) with equation (2) and (3); we have,

$F(u + \delta u, u' + \delta u', x) = F(u, u', x) + \delta F \quad (6)$

And the Taylor series expansion of the variation δF in equation (6) is:

$$F(u + \delta u, u' + \delta u', x) = F(u, u', x) + \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right) + \cdots \quad (7)$$

Hence, the variation that minimizes or maximizes the functional, is determined by the first variation. That is,

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right) dx \qquad (8)$$

Substituting for $\delta u'$ in (8) using equation (4); we get,

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}d(\delta u)\right) dx \qquad (9)$$

Performing integration by part on the second terms of equation (9) and replacing it; we have,

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u - \frac{d}{dx}(\frac{\partial F}{\partial u'})\delta u\right) dx \qquad (10a)$$

Displacement u, minimizes the variation of the total potential (δI) if $u(x) = \bar{u}(x)$; implying, $\delta I = 0$. Bring equation (10) to:

$$\int_a^b \left(\frac{\partial F}{\partial u}\delta u - \frac{d}{dx}(\frac{\partial F}{\partial u'})\right) \delta u\, dx \qquad (10b)$$

On a further simplification of equation (10b); we have,

$$\frac{\partial F}{\partial u}\delta u - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \qquad (11)$$

Equation (11) is the famous Euler Lagrange equation of solid mechanics $(Shehu; 2012)$[6].

2.2 Formulation of Functional

The functional is the mathematical formulation of a solid subjected to pressure load; that is, in its stable equilibrium configuration $(Dass; 2008\,)$[4]. Derived from the analytic definition of elasticity with the external loading parameters. That is,

$$F = {}^1\!/_2\, AE\left(\frac{du}{dx}\right)^2 - qu \qquad (12)$$

Where, A is the surface area of the solid, E is the Young Modulus of the concrete, u is the displacement, and q is the load. And the first term of equation (12) is the strain energy stored in the beam.

The necessary condition for the functional, equation (12) to remain in its stable equilibrium configuration is equivalent to finding a displacement u, that satisfies the governing differential equation representing the functional $(Shehu; 2012)$[8].

2.3 Derivation of the Governing Equation

Using Euler Lagrange equation (11) of solid mechanics to transform the functional equation (12); that is,

$$\frac{\partial F}{\partial u} = -q \qquad (13)$$

$$\frac{\partial F}{\partial u'} = AE\frac{du}{dx} \qquad (14)$$

Substituting equation (13) and (14) into equation (11); we have,

$$-q - \frac{d}{dx}\left(AE\frac{du}{dx}\right) \qquad (15a)$$

$$AE\frac{d^2u}{dx^2} + q = 0 \qquad (15b)$$

In 2-D; that is, in $(x\ and\ y)$ the loading parameter q will cancel out, leaving equation (15b) as;

$$AE\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad (16a)$$

Hence, equation (16) is the governing differential equation representing the functional equation (12) in 2-D.

3. SOLUTION BY FINITE ELEMENT ANALYSIS

$$AE\iint W\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) dA = 0 \qquad (16b)$$

Where W is the weighted residue, also called the interpolation function. Equation (16b) then simplify into the weak form as:

$$\mathrm{AE}\iint\left(\frac{\partial W}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial W}{\partial y}\frac{\partial \psi}{\partial y}\right) dA = \mathrm{AE}\oint W\left(\frac{\partial \psi}{\partial x}l_x + \frac{\partial \psi}{\partial y}l_y\right) ds = \mathrm{AE}\oint W\left(\frac{\partial \psi}{\partial n}\right) ds$$

$$AE\iint\left(\frac{\partial W}{\partial x}\frac{\partial \psi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial \psi}{\partial y}\right)dA = AE\oint W\left(\frac{\partial \psi}{\partial n}\right)ds \qquad (17)$$

$$\left(AE\iint [B]^T[B]dA\right)\{\psi\} = AE\oint [N]^T\left(\frac{\partial \psi}{\partial n}\right)ds \qquad (18)$$

$$[k]\{\psi\} = \{f\} \qquad (19)$$

Where the stiffness matrix,

$$[k] = AE\iint [B]^T[B]dA$$

And the loading force,

$$\{f\} = AE\oint [N]^T\left(\frac{\partial \psi}{\partial n}\right)\mathbf{ds}$$

Then, for the entire finite element mesh; we have,

$$AE\sum_{1}^{NELEM}\left(\iint [B]^T[B]dA\right)\{\psi\} = AE\sum_{1}^{NELEM}\oint [N]^T\left(\frac{\partial \psi}{\partial n}\right)ds \qquad (20)$$

Where $[N]$ and $[N]^T$ are the shape function matrix and its transpose respectively. $[B]$ and $[B]^T$ are the derivatives of the shape function. And $\{\psi\}$ is the displacement vector in the x and y direction.

Note: In the Galerkin formulation, we use the same shape function as the weight function (W=N) by (*Shehu*;2012)[8].

3.1 Computation of Interpolation Function

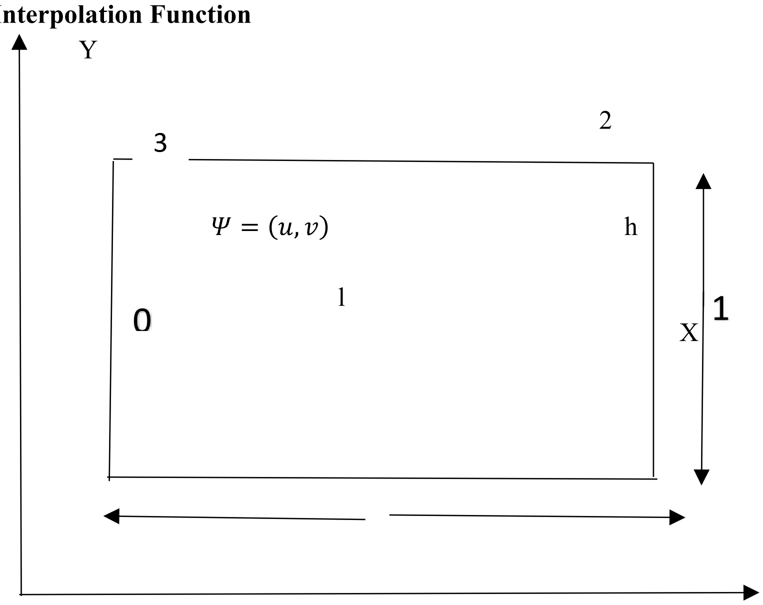

Figure 2: Displacement Distribution in Rectangular Mesh.

Let the axial and transverse displacements in the quadrilateral be represented by u and v respectively.

Here, the unknown field variables vary independently along two directions. Hence, we assume that the displacement field over the mesh is given as:

$$u(x,y) = C_0 + C_1x + C_2y + C_3xy \qquad (21)$$

and

$$v(x,y) = C_0 + C_1x + C_2y + C_3xy \qquad (22)$$

Thus, for the rectangular element of size $(l \times h)$. On computation; we have,

$$u_1 = C_0 \qquad (23)$$

$$u_2 = C_0 + C_1 l \qquad (24)$$

$$u_3 = C_0 + C_1 l + C_2 h + C_3 lh \qquad (25)$$

$$u_4 = C_0 + C_4 h \qquad (26)$$

Solving for, C_o, C_1, C_2, and C_3; we have,

$$C_0 = u_1 \qquad (27)$$

$$C_1 = \frac{u_2 - u_1}{l} \qquad (28)$$

$$C_2 = \frac{u_4 - u_1}{h} \quad (29)$$

$$C_3 = \frac{u_3 + u_1 - u_2 - u_4}{lh} \quad (30)$$

Substituting for, C_o, C_1, C_2, and C_3; in equation (22); we have,

$$u(x,y) = u_1 + \left(\frac{u_2 - u_1}{l}\right)x + \left(\frac{u_4 - u_1}{h}\right)y + \left(\frac{u_3 + u_1 - u_2 - u_4}{lh}\right)xy \quad (31)$$

Expanding and grouping like terms; we get,

$$u(x,y) = \left(1 - \frac{x}{l} - \frac{y}{h} + \frac{xy}{lh}\right)u_1 + \left(\frac{x}{l} - \frac{xy}{lh}\right)u_2 + \left(\frac{xy}{lh}\right)u_3 + \left(\frac{y}{h} - \frac{xy}{lh}\right)u_4 \quad (32)$$

In standard finite element notation equation (32) is written as:

$$u(x,y) = [N_1, N_2, N_3, N_4]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \quad (33)$$

For uniaxial displacement

Similar, for transverse displacement; we have,

$$v(x,y) = [N_1, N_2, N_3, N_4]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} \quad (34)$$

Where

$$N_1 = 1 - \frac{x}{l} - \frac{y}{h} + \frac{xy}{lh}$$

$$N_2 = \frac{x}{l} - \frac{xy}{lh}$$

$$N_3 = \frac{xy}{lh}$$

$$N_4 = \frac{y}{h} - \frac{xy}{lh}$$

Hence, N_i is the shape function, describing the uniform distribution of the displacement field. Since Figure 2; is used to model structural mechanics problems, each node will have two degrees of freedom viz: u and v; hence, we can write the displacement field, using the shape functions derived in equation (33) and (34) as:

$$[N_i]\{\psi\} = [N_i]\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad (35)$$

and

$$\begin{bmatrix} \frac{dN_i}{dx} \\ \frac{dN_i}{dy} \end{bmatrix} = [B] = \begin{bmatrix} \frac{dN_1}{dx} & 0 & \frac{dN_2}{dx} & 0 & \frac{dN_3}{dx} & 0 & \frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_1}{dy} & 0 & \frac{dN_2}{dy} & 0 & \frac{dN_3}{dy} & 0 & \frac{dN_4}{dy} \end{bmatrix}$$

Then;

$$[B]^T[B]=\begin{bmatrix}\frac{dN_1}{dx} & 0\\ 0 & \frac{dN_1}{dy}\\ \frac{dN_2}{dx} & 0\\ 0 & \frac{dN_2}{dy}\\ \frac{dN_3}{dx} & 0\\ 0 & \frac{dN_3}{dy}\\ \frac{dN_4}{dx} & 0\\ 0 & \frac{dN_4}{dy}\end{bmatrix}\begin{bmatrix}\frac{dN_1}{dx} & 0 & \frac{dN_2}{dx} & 0 & \frac{dN_3}{dx} & 0 & \frac{dN_4}{dx} & 0\\ 0 & \frac{dN_1}{dy} & 0 & \frac{dN_2}{dy} & 0 & \frac{dN_3}{dy} & 0 & \frac{dN_4}{dy}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{dN_1}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_4}{dx} & 0\\ 0 & \frac{dN_1}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_4}{dy}\\ \frac{dN_2}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_4}{dx} & 0\\ 0 & \frac{dN_2}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_4}{dy}\\ \frac{dN_3}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_4}{dx} & 0\\ 0 & \frac{dN_3}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_4}{dy}\\ \frac{dN_4}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_4}{dx} & 0\\ 0 & \frac{dN_4}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_4}{dy}\end{bmatrix} \qquad (36)$$

Where

$$\frac{dN_1}{dx}=\frac{y}{l^2}-\frac{1}{l},\qquad \frac{dN_2}{dx}=\frac{1}{l}-\frac{y}{l^2},\qquad \frac{dN_3}{dx}=\frac{y}{l^2},\qquad \frac{dN_4}{dx}=\frac{y}{l^2}$$

And

$$\frac{dN_1}{dy}=\frac{x}{l^2}-\frac{1}{l},\qquad \frac{dN_2}{dy}=-\frac{x}{l},\qquad \frac{dN_3}{dy}=\frac{x}{l^2},\qquad \frac{dN_4}{dy}=\frac{1}{l}-\frac{x}{l^2}$$

Then,

$$\frac{dN_1}{dx}\cdot\frac{dN_1}{dx}=\left(\frac{y}{l^2}-\frac{1}{l}\right)\left(\frac{y}{l^2}-\frac{1}{l}\right)=\frac{y^2}{l^4}-\frac{2y}{l^3}+\frac{1}{l^2}$$

Hence, for $\int_0^l\int_0^1[B]^T[B]dxdy$, computing for each component of the matrix in equation (20) as follows:

$$\int_0^l\int_0^1\frac{dN_1}{dx}\cdot\frac{dN_1}{dx}dxdy=\int_0^l\int_0^1\left(\frac{y^2}{l^4}-\frac{2y}{l^3}+\frac{1}{l^2}\right)dxdy=0.3l^{-1}$$

$$\int_0^l\int_0^1\frac{dN_2}{dx}\cdot\frac{dN_1}{dx}dxdy=\int_0^l\int_0^1\left(\frac{2y}{l^3}-\frac{1}{l^2}+\frac{y^2}{l^4}\right)dxdy=0.3l^{-1}$$

$$\int_0^l\int_0^1\frac{dN_3}{dx}\cdot\frac{dN_1}{dx}dxdy=\int_0^l\int_0^1\left(\frac{y^2}{l^4}-\frac{y}{l^3}\right)dxdy=-0.2l^{-1}$$

$$\int_0^l\int_0^1\frac{dN_4}{dx}\cdot\frac{dN_1}{dx}dxdy=\int_0^l\int_0^1\left(\frac{y^2}{l^4}-\frac{y}{l^3}\right)dxdy=-0.2l^{-1}$$

$$\int_0^l\int_0^1\frac{dN_1}{dy}\cdot\frac{dN_1}{dy}dydx=\int_0^l\int_0^1\left(\frac{x^2}{l^4}-\frac{2x}{l^3}+\frac{1}{l^2}\right)dydx=0.3l^{-1}$$

$$\int_0^l\int_0^1\frac{dN_2}{dy}\cdot\frac{dN_1}{dy}dydx=\int_0^l\int_0^1\left(-\frac{x^2}{l^3}+\frac{x}{l^2}\right)dydx=0.2$$

$$\int_0^l\int_0^1\frac{dN_3}{dy}\cdot\frac{dN_1}{dy}dydx=\int_0^l\int_0^1\left(\frac{x^2}{l^4}-\frac{x}{l^3}\right)dydx=0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_4}{dy} \cdot \frac{dN_1}{dy} dydx = \int_0^l \int_0^1 \left(\frac{2x}{l^2} - \frac{1}{l^2} - \frac{x^2}{l^4}\right) dydx = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_1}{dx} \cdot \frac{dN_2}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{2y}{l^3} - \frac{1}{l^2} - \frac{y^2}{l^4}\right) dxdy = -0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_2}{dx} \cdot \frac{dN_2}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{1}{l^2} - \frac{2y}{l^3} - \frac{y^2}{l^4}\right) dxdy = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_3}{dx} \cdot \frac{dN_2}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y}{l^3} - \frac{y^2}{l^4}\right) dxdy = 0.2l^{-1}$$

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$$\int_0^l \int_0^1 \frac{dN_1}{dy} \cdot \frac{dN_2}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x^2}{l^3} + \frac{x}{l^2}\right) dydx = 0.2$$

$$\int_0^l \int_0^1 \frac{dN_2}{dy} \cdot \frac{dN_2}{dy} dydx = \int_0^l \int_0^1 \left(\frac{x^2}{l^2}\right) dydx = 0.3l$$

$$\int_0^l \int_0^1 \frac{dN_3}{dy} \cdot \frac{dN_2}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x^2}{l^3}\right) dydx = -0.3$$

$$\int_0^l \int_0^1 \frac{dN_4}{dy} \cdot \frac{dN_2}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x}{l^2} + \frac{x^2}{l^3}\right) dydx = -0.2$$

$$\int_0^l \int_0^1 \frac{dN_1}{dx} \cdot \frac{dN_3}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4} - \frac{y}{l^3}\right) dxdy = -0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_2}{dx} \cdot \frac{dN_3}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y}{l^3} - \frac{y^2}{l^4}\right) dxdy = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_3}{dx} \cdot \frac{dN_3}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4}\right) dxdy = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_4}{dx} \cdot \frac{dN_3}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4}\right) dxdy = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_1}{dy} \cdot \frac{dN_3}{dy} dydx = \int_0^l \int_0^1 \left(\frac{x^2}{l^4} - \frac{x}{l^3}\right) dydx = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_2}{dy} \cdot \frac{dN_3}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x^2}{l^3}\right) dydx = -0.3$$

$$\int_0^l \int_0^1 \frac{dN_3}{dy} \cdot \frac{dN_3}{dy} dydx = \int_0^l \int_0^1 \left(\frac{x^2}{l^4}\right) dydx = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_4}{dy} \cdot \frac{dN_3}{dy} dydx = \int_0^l \int_0^1 \left(\frac{x}{l^3} - \frac{x^2}{l^4}\right) dydx = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_1}{dx} \cdot \frac{dN_4}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4} - \frac{y}{l^3}\right) dxdy = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_2}{dx} \cdot \frac{dN_4}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y}{l^3} - \frac{y^2}{y^4}\right) dxdy = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_3}{dx} \cdot \frac{dN_4}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4}\right) dxdy = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_4}{dx} \cdot \frac{dN_4}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4}\right) dxdy = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_1}{dy} \cdot \frac{dN_4}{dy} dydx = \int_0^l \int_0^1 \left(\frac{2x}{l^3} - \frac{1}{l^4} - \frac{x^2}{l^3}\right) dydx = -0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_2}{dy} \cdot \frac{dN_4}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x}{l^2} + \frac{x^2}{l^3}\right) dydx = -0.2$$

$$\int_0^l \int_0^1 \frac{dN_3}{dy} \cdot \frac{dN_4}{dy} dydx = \int_0^l \int_0^1 \left(\frac{x}{l^3} - \frac{x^2}{l^4}\right) dydx = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_4}{dy} \cdot \frac{dN_4}{dy} dydx = \int_0^l \int_0^1 \left(\frac{1}{l^2} - \frac{2x}{l^3} + \frac{x^2}{l^4} \right) dydx = 0.3l^{-1}$$

So that,

$$\int_0^l \int_0^1 [B]^T [B] dxdy = \begin{bmatrix} 0.3l^{-1} & 0 & -0.3l^{-1} & 0 & -0.2l^{-1} & 0 & 0.2l^{-1} & 0 \\ 0 & 0.3l^{-1} & 0 & 0.2 & 0 & 0.2l^{-1} & 0 & -0.3l^{-1} \\ 0.3l^{-1} & 0 & 0.3l^{-1} & 0 & 0.2l^{-1} & 0 & 0.2l^{-1} & 0 \\ 0 & 0.2 & 0 & 0.3l^{-1} & 0 & -0.3 & 0 & -0.2 \\ -0.2l^{-1} & 0 & 0.2l^{-1} & 0 & 0.3l^{-1} & 0 & 0.3l^{-1} & 0 \\ 0 & 0.2l^{-1} & 0 & -0.3 & 0 & 0.3l^{-1} & 0 & 0.2l^{-1} \\ -0.2l^{-1} & 0 & 0.2l^{-1} & 0 & 0.3l^{-1} & 0 & 0.3l^{-1} & 0 \\ 0 & 0.3l^{-1} & 0 & -0.2 & 0 & 0.2l^{-1} & 0 & 0.3l^{-1} \end{bmatrix}$$

$$AE \int_0^l \int_0^1 [B]^T [B] dA = \frac{AE}{l} \begin{bmatrix} 0.3 & 0 & -0.3 & 0 & -0.2 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0.2l & 0 & 0.2 & 0 & -0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2l & 0 & 0.3 & 0 & -0.3l & 0 & -0.2l \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & -0.3l & 0 & 0.3 & 0 & 0.2 \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & -0.2l & 0 & 0.2 & 0 & 0.3 \end{bmatrix}$$

Also, specified in the form of ‘’q’’ per unit length be address in similar manner by performing the integration $\int [N]^T qdx$ and obtain {f}; thus, for a uniformly distributed force q, we can write the equivalent nodal force vector as:(*Ndiwari Ebikiton*, *and Zuonaki Ongodiebi*;2025)[7]

$$\{f\}_C = \int_0^l \int_0^1 [N_i]^T qdxdy = q \begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

And for body forces or gravity loading. Gravity loading as a typical body force is given by ρg per-volume or ρAg per- unit length, where ρ is the mass density of the continuum. The equivalent nodal force vector for the distributed body force can be obtain as:

$$\{f\} = \int_v [N_i]^T l\rho g dxdy = \int_0^l \int_0^1 [N_i]^T l\rho g dxdy$$

That is;

$$\{f\}_B = \int_0^l \int_0^1 [N_i]^T l\rho g dxdy = l\rho g \begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Hence, the numerical equivalence of equation (3.97); that is, $[k]\{\psi\} = \{f\}$ on a unit cell bases is:

$$\frac{AE}{l}\begin{bmatrix} 0.3 & 0 & -0.3 & 0 & -0.2 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0.2l & 0 & 0.2 & 0 & -0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2l & 0 & 0.3 & 0 & -0.3l & 0 & -0.2l \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & -0.3l & 0 & 0.3 & 0 & 0.2 \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & -0.2l & 0 & 0.2 & 0 & 0.3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = l\rho g \begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + q \begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \tag{27}$$

Where, the first term in equation (27) represents the stiffness matrix of the solid, the second term represents the axial and transverse displacement vector, the third term stands for body forces, and the fourth term denotes concentrated or live load.

3.2 Residual Force Formulation

The analytic form of equation (27) is given as:

$$K\psi = F_B + F_C = \{f\} \tag{28}$$

Hence, the total forces capable of initiating some change in a continuum from equation (27) are body force (F_B) and concentrated or contact force (F_C). Here, we introduce the existence of a third force, called the sedentary or residual force (F_R), this force is responsible for the implosive nature of building collapse in Nigeria and other parts of the world. In this research, we concentrate on Concrete and rein-forced concrete, due to its cheapness, availability and dominance of today’s construction industry. Concrete, a solidified mixture of sand, cement and steel carries within itself the components that lead to the generation of residual force. Which as earlier defined, as forces that are inherent in a body in the absence of external force. Hence it is sedentary and potential in form. It emanates from the chemical composition of the structural material, and the environment on which the structure is place. The following are the components of structural materials whose chemical composition give room to the generation of residual force in concrete:

(i) **Sand**: Sand make-up seventy-five percent mixture proportion in concrete. Chemically known as silicon (11) oxide. There are some highly concentrated silicon (11) oxide; such as, opal, chert, and quartzite, used for the manufacture of chalk. These hyper silicon oxides when contain in a mixture with cement tend to prevent proper bond formation. The partial bond formed will interact with water to produce a swelling residue, which will expand and form a mapping crack on the surface of the concrete. These cracks exposes the interior of the concrete that houses the rein-forced steel to rust. When steel rust, it will expand by six percent. This volumetric increment, places a large tensile stress on the surrounding regions of the embedded steel. Thereby breaking the brittle concrete bond around and along the embedded steel domain. All these will undermine the strength, and reduce the stiffness constant of the concrete and its rein-forced counter-part., which can eventually lead to a sudden and complete failure of a structure over a long period of time.

(ii) **Cement**: Cement derive its power to bind from its alkaline content. Chemical reactivity of any substance is measured by their PH-value. The PH-scale is calibrated from 0 to 14. From 0 to 6, acidic. 7 neutral. 8 to 14 basic (alkaline). Cement has a PH-scale of 13.8. This high alkaline content give cement the potency to hold the discrete seventy five percent mixture content of sand into shape, and at the same time exposes concrete to chemical attack from the surrounding environment. Chemicals gases like: Chlorine, Sulphur and Carbon (iv) oxide will readily interact with the alkaline in cement and drop its PH-level. The dropping PH-level initiate bond breakage within the concrete (creating void within the structural lattice of the solid). When the PH-level drops below 10, it can no longer hold the discrete sand grains into shape; thereby dissolving the continuum (models whose matter in the body is continuously distributed and fills the entire region of space they occupy) state of the concrete to rumbles in a swift and complete manner.(*Abdulrahman Fahad Al Fuhaid, and Akbar Niaz*;2022)[1]

These defects cause by the residual force, reduces the stiffness constant of the whole structure. When the stiffness constant is less than the compressive force (from the body forces and the concentrated forces) due to these residual defects, concrete will implode. The sudden and complete mode of failure, points to the existence of voids cause by numerous bond breakage. Hence the mathematics of this residual force will be derive from the Principle of Stationary Total Potential (PSTP). Which gives; the total strain energy, as given in (*Shehu*;2012)[8].

$$U = \int \left[\frac{1}{2}E\varepsilon_x^2\right]Adx - \int \varepsilon_x E\varepsilon_{xo}Adx + \int \varepsilon_x\sigma_{xo}Adx \qquad (30)$$

Where E is the Young modulus, A is the area, ε_x is the strain, ε_{x0} is the initial strain, σ_x is the stress, and σ_{x0} is the initial stress.

And the potential V of the external force is given by:

$$V=-\left(\int aqdx+\sum a_iP_i\right) \qquad (31)$$

Where P_i, is the concentrated force acting at a point i, a_i is its displacement. Hence the total potential of the structure is obtained as in (*Shehu*;2012)[8]

$$\pi_p=U+V \qquad (32)$$

We shall use equation (32), to further our discussion while attempting to derive the residual form of finite element equation.

Let the displacement (a) at any point P within, be interpolated from the nodal values of displacement {δ} using the shape function N as follows:

$$a=[N]\{\delta\} \qquad (33)$$

Where

$$\{\delta\}=\begin{Bmatrix}u_1\\u_2\end{Bmatrix} \qquad (34)$$

And

$$[N]=\left[\left(1-\frac{x}{l}\right)\quad\left(\frac{x}{l}\right)\right] \qquad (35)$$

The non-zero strain at a point P within the element ε_x is given in terms of the derivative of the displacement field as:

$$\varepsilon_x=[\partial](\alpha)=[\partial][N]\{\delta\}=[B]\{\delta\} \qquad (36)$$

Where $[\partial]$ is a differential operator, and $[B]=[\partial][N]$; we can then, rewrite equation (30) and (31) respectively as:

$$U=\int\frac{1}{2}\{\delta\}^T[B]^T[B]\{\delta\}EAdx-\int\{\delta\}^T[B]^TE\varepsilon_{xo}Adx+\int\{\delta\}^T[B]^T\sigma_{xo}Adx \qquad (37)$$

And

$$V=-\int\{\delta\}^T[N]^Tqdx-\sum\{\delta\}^TN_i^TP_i \qquad (38)$$

Hence, equation (32) takes the form,

$$\pi_P=\int\frac{1}{2}\{\delta\}^T[B]^T[B]\{\delta\}EAdx-\int\{\delta\}^T[B]^TE\varepsilon_{xo}dx+\int\{\delta\}^T[B]^T\sigma_{xo}Adx-\int\{\delta\}^T[N]^Tqdx-\sum\{\delta\}^TN_i^TP_i$$

$$\pi_P=\{\delta\}^T\int\frac{1}{2}[B]^T[B]\{\delta\}EAdx-\{\delta\}^T\int[B]^TE\varepsilon_{xo}dx+\{\delta\}^T\int[B]^T\sigma_{xo}Adx-\{\delta\}^T\int[N]^Tqdx-\{\delta\}^T\sum[N]_i^TP_i \qquad (39)$$

From equation (39), the stiffness matrix [K] and the force vector {f} is deduced as follows:

$$[K]=\int[B]^T[B]EAdx \qquad (40)$$

$$\{f\}=\int[B]^T E\varepsilon_{xo}dx+\int[B]^T\sigma_{xo}\,Adx-\int[N]^T\,qdx-\sum[N]_i^T P_i \quad (41)$$

Then we can rewrite equation (39) as:

$$\pi_P=\frac{1}{2}\int(\{\delta\}^T[K]\{\delta\})\,dx\ -\int(\{\delta\}^T\{f\})dx \quad (42)$$

$$\frac{\pi_P}{\{\delta\}^T}=\frac{1}{2}\int([K]\{\delta\})\,dx-\int(\{f\})\,dx \quad (43)$$

Differentiating equation (43); we have,

$$\frac{\partial}{\partial x}\left(\frac{\pi_P}{\{\delta\}^T}\right)=\frac{1}{2}[K]\{\delta\}-\{f\} \quad (44)$$

Hence, the variation of the total potential of the structure π_P, is cause by the residual factors prevalent in the structural material and the environment the structure is place.

That is,

$$\frac{\partial\pi_P}{\partial x\{\delta\}^T}=F_R \quad (45)$$

Leaving equation (44);

$$F_R=\frac{1}{2}[K]\{\delta\}-\{f\}$$

$$\{f\}+F_R=\frac{1}{2}[K]\{\delta\}$$

If there is no variation of the total potential, $\frac{\partial\pi_P}{\partial\{\delta\}^T}=0$, and $K\psi=F_B+F_C=\{f\}$

$$F_B+F_C+F_R=\frac{1}{2}[K]\{\delta\}=\frac{1}{2}[K]\{\psi\}=$$

$$\frac{AE}{2l}\begin{bmatrix} 0.3 & 0 & -0.3 & 0 & -0.2 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0.2l & 0 & 0.2 & 0 & -0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2l & 0 & 0.3 & 0 & -0.3l & 0 & -0.2l \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & -0.3l & 0 & 0.3 & 0 & 0.2 \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & -0.2l & 0 & 0.2 & 0 & 0.3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}=$$

$$\frac{AE}{l}\begin{bmatrix} 0.15 & 0 & -0.15 & 0 & -0.1 & 0 & 0.1 & 0 \\ 0 & 0.15 & 0 & 0.1l & 0 & 0.1 & 0 & -0.15 \\ 0.15 & 0 & 0.15 & 0 & 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1l & 0 & 0 & 0 & -0.3l & 0 & -0.2l \\ -0.1 & 0 & 0.1 & 0 & 0.15 & 0 & 0.15 & 0 \\ 0 & 0.1 & 0 & -0.15l & 0 & 0.15 & 0 & 0.1 \\ -0.1 & 0 & 0.1 & 0 & 0.15 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & -0.1l & 0 & 0.1 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad (46)$$

3.3 Modeling and Simulation

3.3.1 Modeling of a Concrete Beam

Step by step approach for modeling of a concrete beam

Open Abaqus CAE working Environment.

Go to: File>New Model Database>Click on Standard Explicit

Go to: Model 1; Right Click to Rename>concretebeam>Click Ok.

Go to: Parts; Double Click, Name your part:

Concretebeam

Click 3D

Click Deformable

Click Solid & Extrude

Approximate Size: 5

Click Continue

Click on Line Icon

Start at (0,0) and Sketch your Beam Model
Click on equal Length Constraint
Click Dimension to add Dimension: Length 10, Width 0.2, Depth 0.1
Click Ok, to Complete Model Creation.
Go to: Material, Double Click to have the Material Window
Name: concretebeam
General>Density: 2400
Mechanical>Elasticity>Click Elastic
Young Modulus: 55E3, Poisson Ratio: 0.2
Click Ok.
Go to: Section. Double Click
Name: concretebeamsection
Click Solid and Homogeneous
Click Continue and Ok.
Go to: Part Container
Click on concretebeam Container
Double Click on Section Assignment
Click on the Created Model to Assign Section
Click Ok. Model color green indicating section assignment
Go to: Assembly, Double Click
It blank-out the Model
Expand the Assembly Container
Double Click Instances
Click Dependent (mesh the part)
Click Ok.
Go to: Steps, Double Click
Type in Loadingstep
Click Initial
Click General
Click Static General
Click Continue
Time Period: 1
Click Ok
Go to: Field Output Request
Click Preselected Default
Click Ok
Go to: History Output Request
Click Preselected Default
Click Ok.
Go to: Load. Double Click
Name: Loadingstep
Click Mechanical>Pressure load
Select the Surface to be Loaded (centre)
Load Magnitude: 240
Click Ok. (Arrows popp-up, indicating load applied)
Go to: Boundary Condition. Double Click
Name: fixedatbothends
Click Mechanical>Symmetry/Encastre
Click Continue
Select the Region for the BCs
Click Done
Click Encastre
Continue Ok.
Go to: Part Container

Click on Mesh empty (Double Click)
Go to: Mesh at the top Icon
Click Element type
Click Standard
Click Linear
Click Ok
Select Seeding the Part
Approximate Size: 0.1
Click Ok and Done
Click Yes to Complete the Meshing
Go to: Job, Double Click
Name: Loadedconcretebeam
Click Model
Click Full Analysis
Click Ok
Go to: Job Container
Click loadedconcretebeam>Right Click>Click Submit
The processor (solver) will run and indicate completed
Right Click on loadedconcretebeam>Click Result

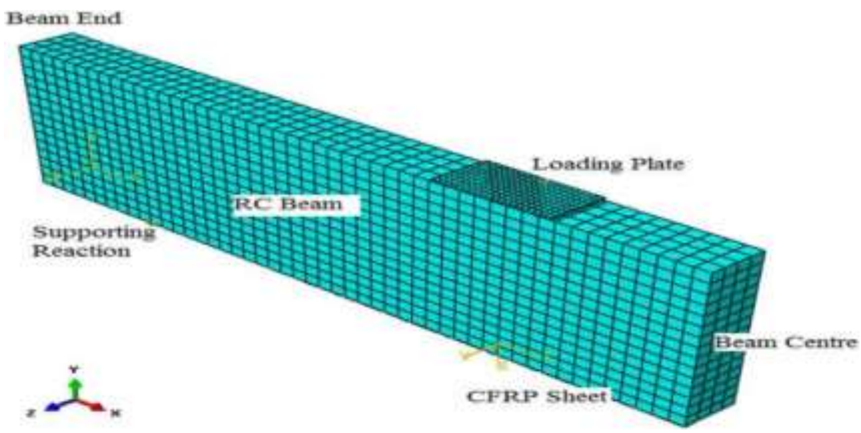

Figure 3: Showing the Discretized Concrete Beam Model

3.4.2 Simulation of the Model

Simulation in Abaqus CAE is carried out in the post processing phase as follows:
Click loadedconcretebeam>Right Click>Click Submit
The processor (solver) will run and indicate completed
Right Click on loadedconcretebeam>Click Result
Go to: Result Module> Click on Deform Icon
It will display the following Contour Plot on viewing port

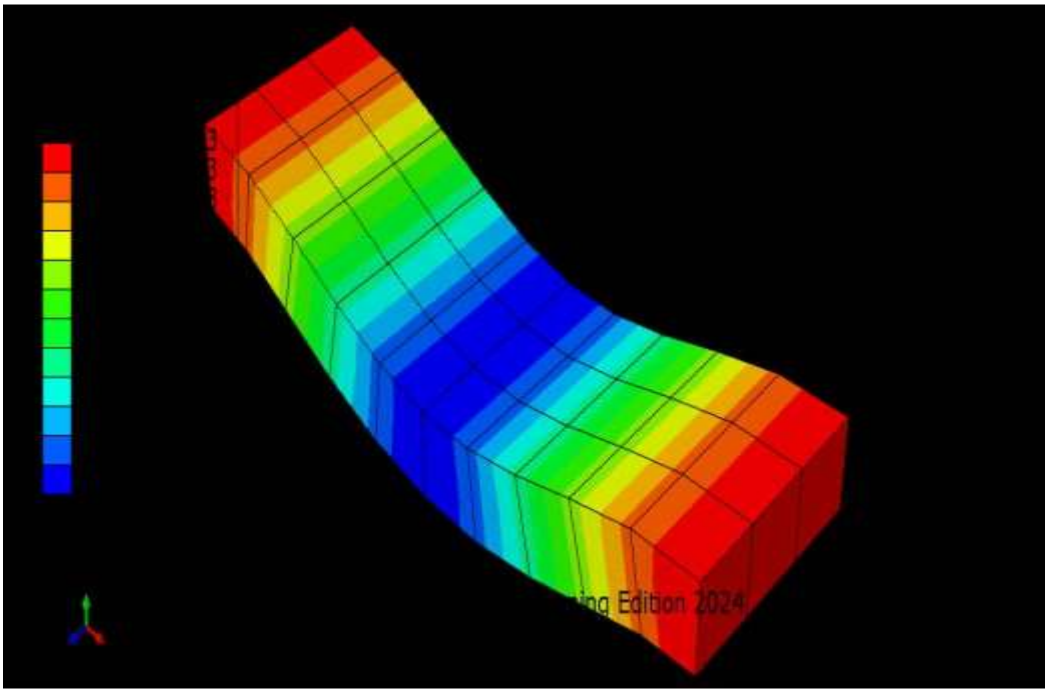

Figure 4: Contour Plot of Concrete Beam Model

To plot XY graph in Abaqus CAE, we create XY-Data by:
Go to: Tools > XY-Data > Create
Select ODB history output
Select (Force/Displacement, and Stress/Strain)
Click: Save as to save the XY-Data Object
Click: Plot to display the following tables and graphs:

Table 1: XY-Data for Force versus Displacement

Force	0	20	40	60	80	100	120	140	160	180	200	220
Displacement	0	1	2	3	4	5	6	8	9	10	11	12

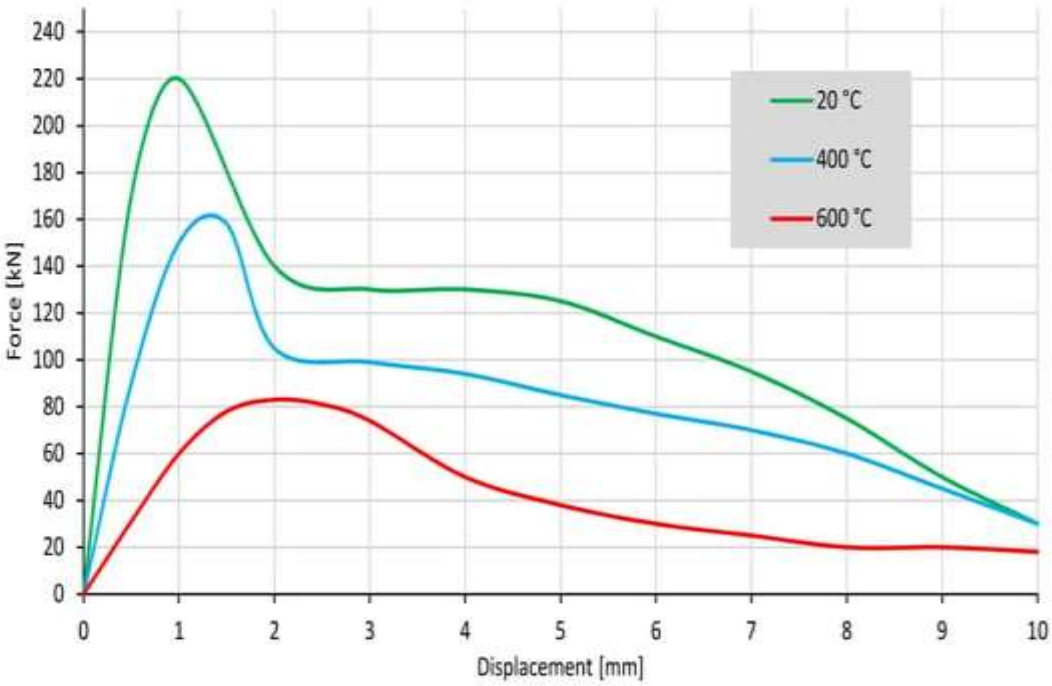

Figure 5: Force versus Displacement Graph

Table 2: XY-Data for Stress versus Strain (Concrete Compressive Behavior)

Stress	0	10	20	30	40	50	60
Strain	0	0.001	0.002	0.003	0.004	0.005	0.006

Table 3: XY-Data for Stress versus Strain (Concrete Tensile Behavior)

Stress	0	0.5	1.0	1.5	2.0	2.5	3.0
Strain	0	0.001	0.002	0.003	0.004	0.005	0.006

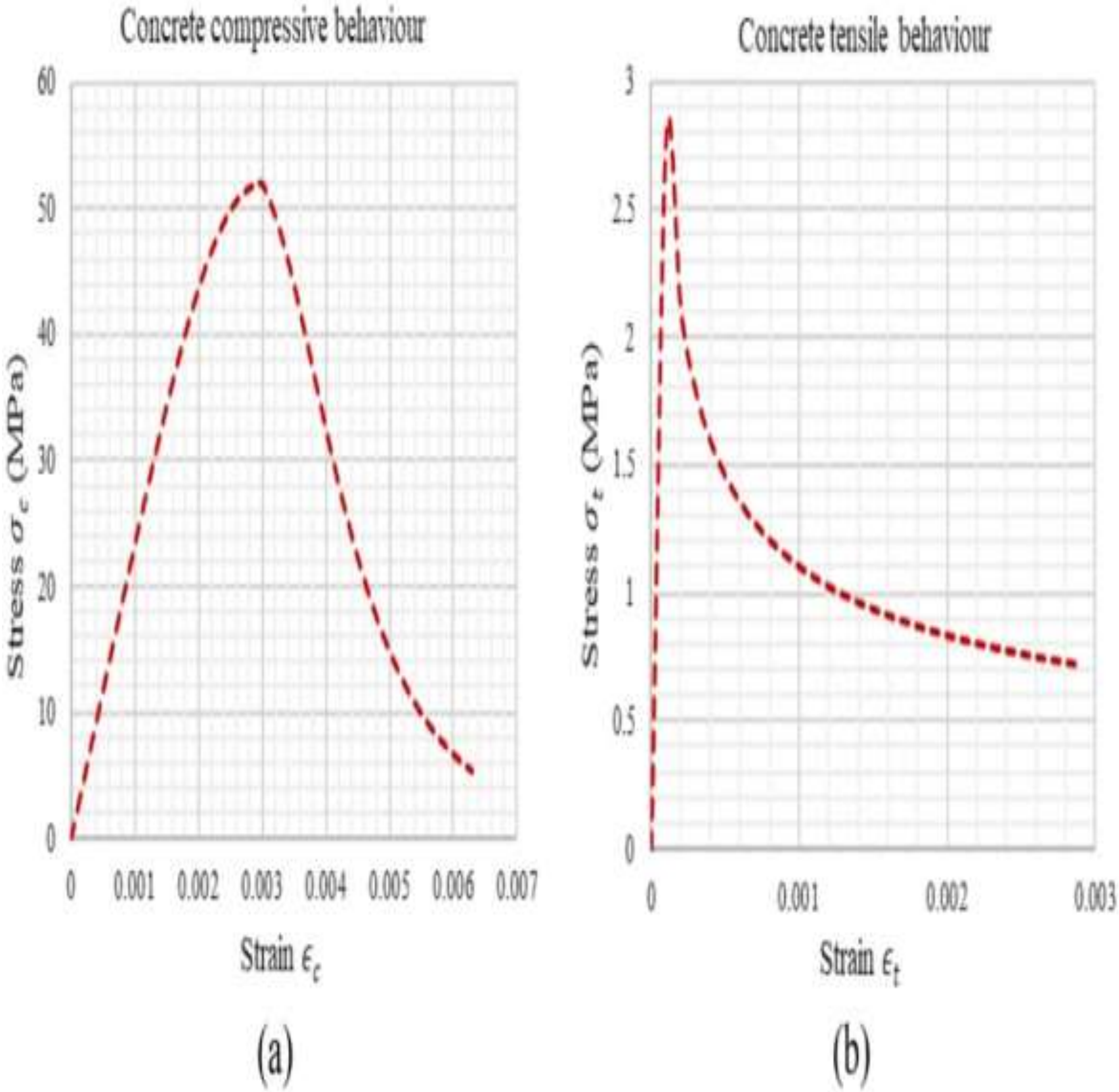

Figure 6: Graph of Stress versus Strain

4. RESULT AND DISCUSSION

4.1 Results

Analytic results from the deformation analysis.

$$\frac{\mathrm{AE}}{\mathrm{l}}\begin{bmatrix} 0.3 & 0 & -0.3 & 0 & -0.2 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0.2\mathrm{l} & 0 & 0.2 & 0 & -0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2\mathrm{l} & 0 & 0.3 & 0 & -0.3\mathrm{l} & 0 & -0.2\mathrm{l} \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & -0.3\mathrm{l} & 0 & 0.3 & 0 & 0.2 \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & -0.2\mathrm{l} & 0 & 0.2 & 0 & 0.3 \end{bmatrix}\begin{Bmatrix} \mathrm{u}_1 \\ \mathrm{v}_1 \\ \mathrm{u}_2 \\ \mathrm{v}_2 \\ \mathrm{u}_3 \\ \mathrm{v}_3 \\ \mathrm{u}_4 \\ \mathrm{v}_4 \end{Bmatrix} = \mathrm{l}\rho\mathrm{g}\begin{bmatrix} \frac{4\mathrm{l}-3}{4} \\ \frac{4\mathrm{l}-3}{4} \\ \frac{2\mathrm{l}-1}{4} \\ \frac{2\mathrm{l}-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + \mathrm{q}\begin{bmatrix} \frac{4\mathrm{l}-3}{4} \\ \frac{4\mathrm{l}-3}{4} \\ \frac{2\mathrm{l}-1}{4} \\ \frac{2\mathrm{l}-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \quad (4.1)$$

$$\mathrm{F_B} + \mathrm{F_C} + \mathrm{F_R} = \frac{1}{2}[\mathrm{K}]\{\delta\} = \frac{1}{2}[\mathrm{K}]\{\psi\} =$$

$$\frac{\mathrm{AE}}{\mathrm{l}}\begin{bmatrix} 0.15 & 0 & -0.15 & 0 & -0.1 & 0 & 0.1 & 0 \\ 0 & 0.15 & 0 & 0.1\mathrm{l} & 0 & 0.1 & 0 & -0.15 \\ 0.15 & 0 & 0.15 & 0 & 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1\mathrm{l} & 0 & 0 & 0 & -0.3\mathrm{l} & 0 & -0.2\mathrm{l} \\ -0.1 & 0 & 0.1 & 0 & 0.15 & 0 & 0.15 & 0 \\ 0 & 0.1 & 0 & -0.15\mathrm{l} & 0 & 0.15 & 0 & 0.1 \\ -0.1 & 0 & 0.1 & 0 & 0.15 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & -0.1\mathrm{l} & 0 & 0.1 & 0 & 0.15 \end{bmatrix}\begin{Bmatrix} \mathrm{u}_1 \\ \mathrm{v}_1 \\ \mathrm{u}_2 \\ \mathrm{v}_2 \\ \mathrm{u}_3 \\ \mathrm{v}_3 \\ \mathrm{u}_4 \\ \mathrm{v}_4 \end{Bmatrix} \quad (4.2)$$

Table 4.1: XY-Data for Force versus Displacement

Force	0	20	40	60	80	100	120	140	160	180	200	220
Displacement	0	1	2	3	4	5	6	8	9	10	11	12

Table 4.2: XY-Data for Stress versus Strain (Concrete Compressive Behavior)

Stress	0	10	20	30	40	50	60
Strain	0	0.001	0.002	0.003	0.004	0.005	0.006

Table 4.3: XY-Data for Stress versus Strain (Concrete Tensile Behavior)

Stress	0	0.5	1.0	1.5	2.0	2.5	3.0
Strain	0	0.001	0.002	0.003	0.004	0.005	0.006

4.2: Discussion

Equation (4.1) is a mathematical model describing a solid in a stable equilibrium configuration. Generated from one finite element mesh. Is a symmetric matrix of 8 by 8, with 64 entries. The upper-lower triangular entries are all equal, indicating balance within the structural lattice of the continuum. And the eigen-values are all 0.3, conforming with the continuity condition in calculus; that is, the left hand limit, the right hand limit, and the value of the function at the point are the same. Which is also in line with the general assumption in finite element analysis, that solids are connected and held into shape by their nodal connectivity; a state, where the solid undergoes rigid body displacement. Which is a form of displacement that cannot cause deformation because the atomic and sub-atomic particles in the continuums are in conformity with Hooke’s law; which states that, “displacement is directly proportional to the existing forces exerted on the solid. In essence, the solid is within its elastic region.

While equation (4.2) is the mathematical model that captures the phenomenon of implosive deformable residual force. Implosion as an inward explosion is cause by pressure difference between two opposing forces. In this case under consideration, compressive forces versus resistive force. Compressive forces are generated by contact and body forces. While the resistive force emanates from the stiffness constant of concrete, that ranges between 10 to 60 Mpa; also known as, the Young Modulus. Cement, the binding agent of concrete drops its potency as a result of its interaction with atmospheric gases such as carbon dioxide. This interaction is shown in equation (4.2) in the reduction of the stiffness matrix by half. The zero entries at the principal diagonal of the stiffness matrix in equation (4.2) reveals discontinuities within the structural lattice of the entire concrete. This discontinuities reduces the resistive force. When the resistive force drops below the opposing compressive forces, the structure implodes. This is the secret behind structures, burying their occupants during failure. The entire building is weaken below her resistive capacity due to massive bond breakage cause by carbonation.

Table 4.1, shows the relationship between force and displacement. From the curves generated by Table 4.1; that is Figure 5, the displacements peaked at 2mm, 1.5mm, and 1mm corresponding to a force of 85KN, 160KN, and 220KN respectively. The displacement is maximum at the fixed ends of the beam. Indicated with a red curve. At a force of 85KN, the material reached its elastic limit of 2mm. Implying, concrete displays linear elasticity at the range of 1KN to 85KN. Any force beyond this magnitude will overwhelm the structural material, leading to permanent deformation, as shown in Figure 5, by the dropping of the curve. And the heat dissipated at the transformation point is 600 degree Celsius. At that temperature, the sub-atomic particles undergoes dislocation.

Table 4.2, and 4.3, expresses the stress-strain relationship in their compressive and tensile form. Figure 6(a), shows the stress-strain curve of concrete under compressive loading. At a peak force intensity of 52Mpa, concrete undergo a deformation of 0.003. While at a stress of just 2.5Mpa concrete undergo deformation as a tensile response to load. From these information deduced from Figure 6 (a & b), concrete is more tolerant to compressive loading than tensile stress. Hence, to enhance tensile performance of concrete, steel reinforcement should be embedded in concrete.

Also, the curve in Figure 5, generated from Table 4.1, categorizes the material behavior of concrete into two state: elastic region and plastic region. These two states are also contained in the mathematics model of our stiffness matrix of equation (4.2). As continuous and discontinuous entries at the principal diagonal of the stiffness matrix.

The elastic region on the graph is represented by the 0.15 entries on the stiffness matrix. While the plastic (deformed) region is expressed by the 0.00 entries. And the maximum loading force on the graph in Figure 5, is 85KN matching the stiffness matrix resistive equivalence of equation (4.1); hence, ensuring a stable equilibrium configuration in the solid. While equation (4.2), reduces the stiffness matrix by half; thereby, amplifying the 85KN compressive force beyond the resistive capacity of the concrete, leading to the sudden and complete failure of structures termed (implosion).

4.3 Conclusion

Deformation is the change in the position of a particle within a solid from the initial configuration to its current configuration. Caused by the variation in the displacement field within the solid. Deformation is analyzed in a concrete beam here, through the field variables, (displacement, stress, and strain) using two approaches (finite element mathematics, and finite element solver). The analytic result obtained as stiffness matrices, through finite element mathematics is compared the result obtained from finite element solver (Abaqus). The simulated curve shows two states of the beam under force: as elastic and plastic state; while, the stiffness matrices, reveals the continuous and discontinuous states. Both represent deformations , as charts and figures.

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