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Formulation and Implementation of Obreckoff's-Type Optimized Hybrid Block Method for the Solution of Second Order Ordinary Differential Equation Using Free Parameter Off-Grid Points

Dominic Raymond[1], John Dan-Ndari[2], Olurotimi Samuel Ebenezer[3], Tsoke Peter Pantuvo[4]
[1,2,4]Department of Mathematics, Federal University, Wukari, Nigeria
[3]Department of Mathematics, Federal College of Education, Obudu, Nigeria

ABSTRACT: This paper examines the formulation and development of Obrechkoff-type three-step optimized block schemes for solving second-order initial value problems (IVPs) of ordinary differential equations. In the derivation of the method, the assumed power series is interpolated at x_n and x_{n+1} while the second derivative is collocated at all grid and off-grid points in the interval of consideration. The method's relevant properties were examined and found to be zero stable, consistent and convergent. The application of the method to some second-order initial value problems shows the derived method performed better in terms of accuracy and effectiveness; outperforming some existing methods when compared.

KEYWORDS: Collocation techniques, Optimized hybrid points, Initial value problems, Implicit hybrid method,

I. INTRODUCTION

This paper considers an effective solution to the initial value problems of the second-order differential equation of the form

$\Upsilon''(x) = \varrho(x, \Upsilon, \Upsilon'),$ (1)

subject to the initial conditions

$\Upsilon(x_0) = y_0, \Upsilon'(x_0) = \Upsilon_0',$ where $x \in [a, b]$ and f is continuous in $[a, b]$.

The general Obrechkoff's formula, which is given in the form

$\Upsilon_{n+k} = \sum_{j=0}^{k-1} \mu_j \Upsilon_{n+j} + \sum_{i=2} h^{i} \sum_{j=0}^{k} \varphi_j \varrho_{n+j}^{(i)} + \varphi_v \varrho_{n+v}^{(i)}$ (2)

is employed. Equation (2) has the advantage of reaching higher-order with few functions to be evaluated per step [19].

Problems such as in (1) above frequently arise in fields of physical sciences and engineering, e.g. structural mechanics, fluid dynamics, control theory, waves and vibration analysis [16]. Finding suitable solutions to second-order IVPs is usually challenging necessitating the need for numerical methods to provide suitable approximate solutions to such [16], [9]. In the past, use of reduction method to solve second-order ordinary differential equations was popular among researchers [16], [18]. This approach involves reducing the differentials to its corresponding system of first order ODEs, then applying appropriate numerical approach to solve the system of first-order equation obtained. This approach has resulted in significant computational challenges and inefficient utilization of computer resources, hence, affecting the accuracy of the method in terms of error. The method has been extensively analyzed in [16] and [18].

Other schemes developed by scholars to handle the solution of second-order IVPs include the predictor-corrector method [3] and Taylors series [9] which have similar disadvantages as the reduction method. Some researchers applied direct methods to solve (1) which include Kamarun *et al* [11], Raymond *et al* [7], Jator & Li *et al* [14], Areo *et al* [17], Adeyeye & Omar [7], Akinnukawe & Okunuga [1], James *et al* [4], Ajinnuhi *et al* [5], Hussain *et al* [6], and lots more; but the accuracy of their methods can be in terms of error improved upon.

The intent of this study is to formulate and implement a new scheme for the effective solution of (1) using optimized hybrid points q_1 and q_2 such that $0 < q_1 < q_2 < 1 < 2 < 3$ with the aim to produce an approximation with reduced error compared with some existing methods in literature.

II. DERIVATION OF THE ONE-STEP OPTIMIZED HYBRID METHODS

Let the approximate solution to (1) be a power series of the form

$\Upsilon(t) = g(x) \approxeq \sum_{j=0}^{s} \rho_j x^j$ (3)

where $\rho \in R$ are real unknown coefficients that will be determined.

The second derivative of Equation (3) gives

$\Upsilon''(x) = g''(x) = \sum_{j=2}^{s-1} \rho_j j(j-1)x^{j-2} = \varrho(x, \Upsilon, \Upsilon')$ (4)

Interpolating equation (3) at x_n and x_{n+1} and collocating equation (4) at all points $x_{n+m} = x_n + mh,\ m =$

$\{0, q_1, q_2, 1,2,3\}$, where q_1 and q_2 are the two hybrid points. In this way we obtain the system of nonlinear equations of the form

$$V = MD \qquad (5)$$

Where $M = \begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_7 \end{bmatrix}$, $\quad V = \begin{bmatrix} \Upsilon_n \\ \Upsilon'_n \\ \varrho_n \\ \varrho_{n+q_1} \\ \varrho_{n+q_2} \\ \varrho_{n+1} \\ \varrho_{n+2} \\ \varrho_{n+N} \end{bmatrix}$

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 & (x_n + h)^5 & (x_n + h)^6 & (x_n + h)^7 \\ 0 & 0 & 2 & 6\,x_n & 12\,x_n^2 & 20\,x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 2 & 6\,h\,q_1 + 6\,x_n & 12\,(h\,q_1 + x_n)^2 & 20\,(h\,q_1 + x_n)^3 & 30\,(h\,q_1 + x_n)^4 & 42\,(h\,q_1 + x_n)^5 \\ 0 & 0 & 2 & 6\,h\,q_2 + 6\,x_n & 12\,(h\,q_2 + x_n)^2 & 20\,(h\,q_2 + x_n)^3 & 30\,(h\,q_2 + x_n)^4 & 42\,(h\,q_2 + x_n)^5 \\ 0 & 0 & 2 & 6\,x_n + 6\,h & 12\,(x_n + h)^2 & 20\,(x_n + h)^3 & 30\,(x_n + h)^4 & 42\,(x_n + h)^5 \\ 0 & 0 & 2 & 6\,x_n + 12\,h & 12\,(x_n + 2h)^2 & 20\,(x_n + 2h)^3 & 30\,(x_n + 2h)^4 & 42\,(x_n + 2h)^5 \\ 0 & 0 & 2 & 6\,x_n + 18\,h & 12\,(x_n + 3h)^2 & 20\,(x_n + 3h)^3 & 30\,(x_n + 3h)^4 & 42\,(x_n + 3h)^5 \end{bmatrix}$$

Using Gaussian elimination method, to solve (4) gives the coefficient ρ_i's for i=0(1)7.

The obtained values of these coefficients, are then substituted into (3) to give a continuous implicit hybrid method of the form;

$$\begin{aligned} g(x_n + mh) \quad &= g(x) \\ &= \rho_0 \Upsilon_n + h\rho_1 \Upsilon'_n \\ &+ h^2\big[\rho_2 \varrho_n + \rho_3 \varrho_{n+q_1} \\ &+ \rho_4 \varrho_{n+q_2} \ + \rho_5 \varrho_{n+1} \\ &+ \rho_6 \varrho_{n+2} + \rho_7 \varrho_{n+3}\big] \end{aligned}$$

(6)

By substituting 1 for m in (6), we obtain a set of multistep formulas to approximate the solutions of (1) at the points x_{n+1}, which yields

$$\begin{aligned} &x_{n+1} \\ &= \Upsilon_n + h\Upsilon'_n \\ &+ h^2 \Big[\frac{1}{2520} \frac{-147q_1 - 147q_2 + 679q_1q_2 + 53}{q_1q_2} \varrho_n \\ &+ \frac{1}{420} \frac{147q_2 - 53}{q_1(q_1 - 3)(q_1 - 1)(q_1 - 2)(q_1 - q_2)} \varrho_{n+q_1} \\ &- \frac{1}{420} \frac{147q_1 - 53}{q_2(q_2 - 3)(q_2 - 1)(q_2 - 2)(q_1 - q_2)} \varrho_{n+q_2} \qquad (7) \\ &+ \frac{1}{840} \frac{-119q_1 - 119q_2 + 266q_1q_2 + 66}{(q_2 - 1)(q_1 - 1)} \varrho_{n+1} \\ &- \frac{1}{840} \frac{-35q_1 - 35q_2 + 91q_1q_2 + 17}{(q_2 - 2)(q_1 - 2)} h^2 \varrho_{n+2} \\ &+ \frac{1}{2520} \frac{-21q_1 - 21q_2 + 56q_1q_2 + 10}{(q_2 - 3)(q_1 - 3)} \ h^2 \varrho_{n+3}\Big] \end{aligned}$$

By differentiating Equation (6), a multistep formula to approximate the solution of Equation (1) at the point x'_{n+1} is obtained as

$$\begin{aligned} hx'_{n+1} &= \Upsilon'_n + \frac{1}{360} \frac{\%-38q_1-38q_2+135q_1q_2+17}{q_1q_2} h\varrho_n \\ &+ \frac{1}{60} \frac{\%38q_2-17}{q_1(q_1-3)(q_1-1)(q_1-2)(q_1-q_2)} h\varrho_{n+q_1} \\ &- \frac{1}{60} \frac{38q_1-17}{q_2(q_2-3)(q_2-1)(q_2-2)(q_1-q_2)} h\varrho_{n+q_2} \\ &+ \frac{1}{120} \frac{-57q_1-57q_2+95q_1q_2+40}{(q_2-1)(q_1-1)} h\varrho_{n+1} \qquad (8) \\ &- \frac{1}{120} \frac{-12q_1-12q_2+25q_1q_2+7}{(q_2-2)(q_1-2)} h\varrho_{n+2} \\ &+ \frac{1}{360} \frac{-7q_1-7q_2+15q_1q_2+4}{(q_2-3)(q_1-3)} h\varrho_{n+3} \end{aligned}$$

To obtain the values of the optimized points, we expand (7) using Taylor series around the point x_{n+1} to obtain the corresponding local truncation error as

$$\mathcal{L}(\Upsilon(x_{n+1}), h) = \frac{-(-106q_1-106q_2+294q_1q_2+49)\mu\Upsilon^{(8)}(x_n)h^{(8)}}{604\,800} + 0(h^9) \qquad (9)$$

Equation (9) was then optimized by equating it to zero, taking q_2 as a free parameter by assigning the value $q_2 = \frac{2}{3}$ in (9) we obtain $q_1 = \frac{13}{54}$. Substituting the values of q_1 and q_2 into equation (5), and solving using Gaussian elimination method, we obtain the block form of the equation in form of

$$A^{[0]}\Upsilon_m^{[1]} = A^{[1]}\Upsilon_m^{[0]} + \sum_{i=0}^{k} D^{[i]}\, G_m^{[i]}, \quad i = \{0, q_1, q_2, 1, 2, N\} \qquad (10)$$

written explicitly as

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} \Upsilon_{n+\frac{13}{54}}\\ \Upsilon_{n+\frac{2}{3}}\\ \Upsilon_{n+1}\\ \Upsilon_{n+2}\\ \Upsilon_{n+3}\\ \Upsilon'_{n+\frac{13}{54}}\\ \Upsilon'_{n+\frac{2}{3}}\\ \Upsilon'_{n+1}\\ \Upsilon'_{n+2}\\ \Upsilon'_{n+3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{13}{54}h\\ 1 & \frac{2}{3}h\\ 1 & h\\ 1 & 2h\\ 1 & 3h\\ 0 & 1\\ 0 & 1\\ 0 & 1\\ 0 & 1\\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Upsilon_n\\ \Upsilon'_n \end{bmatrix} + \begin{bmatrix} \frac{164964973069}{10413862744320}h^2\\ \frac{47903}{995085}h^2\\ \frac{4631}{65520}h^2\\ \frac{109}{4095}h^2\\ \frac{2211}{7280}h^2\\ \frac{5083479583}{55099802880}h\\ \frac{994}{15795}h\\ \frac{113}{1560}h\\ -\frac{38}{195}h\\ \frac{483}{520}h \end{bmatrix} [\varrho_n] + \begin{bmatrix} \frac{12213887}{405798912000} & \frac{8912896}{1728003375} & \frac{65536}{2370375}\\ -\frac{1058841}{184683593728} & -\frac{46}{365715} & \frac{9}{2408}\\ \frac{3411821}{2415919104000} & \frac{304}{5740875} & -\frac{13}{63000}\\ \frac{2794421}{4227072000} & -\frac{13462407}{115427246080} & \frac{957999}{33554432000}\\ \frac{131072}{3918375} & \frac{5}{3483} & -\frac{4}{91125}\\ \frac{4096}{37625} & \frac{99}{3440} & -\frac{1}{750}\\ \frac{250243}{22016000} & -\frac{1577457}{901775360} & \frac{333739}{786432000}\\ \frac{1441792}{9142875} & \frac{52}{1935} & -\frac{92}{30375}\\ \frac{32768}{112875} & \frac{261}{1720} & \frac{1}{500}\\ \frac{57743}{412800} & -\frac{175959}{11272192} & \frac{36701}{9830400}\\ \frac{262144}{609525} & \frac{29}{129} & -\frac{44}{2025}\\ \frac{8192}{22575} & \frac{81}{172} & \frac{8}{75} \end{bmatrix} \begin{bmatrix} \varrho_{n+\frac{13}{54}}\\ \varrho_{n+\frac{2}{3}}\\ \varrho_{n+1}\\ \varrho_{n+2}\\ \varrho_{n+3} \end{bmatrix} \qquad (11)$$

Equations (11) forms the Three-step Optimized Block Scheme (TOBS) derived for the approximation of second-order initial value problems (IVPs) of (1).

III. ANALYSIS OF BASIC PROPERTIES OF TOBS

A. *Order of the Block*

Let the linear difference operator $\mathcal{L}$ associated with the new method be defined by

$$\mathcal{L}[(\Upsilon(x), h)] = \sum_{j=0}^{N} \rho_j \Upsilon(x+jh) - h^2 \sum_{j=0}^{N} \left(\varrho_j \Upsilon''(x+jh)\right) \qquad (12)$$

Equation (12) can be written in Taylor's series expansion about the point x_n to obtain the expression

$$\mathcal{L}[(\Upsilon(x), h)] = c_0\Upsilon(x) + \bar{c}_1 h\Upsilon'(x) + \bar{c}_2 h^2\Upsilon''(x) + \cdots + \bar{c}_{p+2}h^{p+2}\Upsilon^{(p+2)}(x) + \ldots. \qquad (14)$$

Similarly, the new optimized second order hybrid methods (TOBS) is of order p if,

$$\mathcal{L}[(\Upsilon(x), h)] = \mathcal{O}(h^{p+2}), \quad \bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \cdots = \bar{c}_{p+1} = 0,\ \bar{c}_{p+2} \neq 0$$

Therefore, the principal local truncation error $x_n + N$ is then defined to be

$$\bar{c}_{p+2}h^{p+2}\upsilon^{(p+2)}(x_n)$$

Hence, $c_{p+2} = 8$ in which $c_0 = c_1 = c_2 = c_3 = c_4 = \cdots = c_7$. The new method has order $p = (6,6,7,6,6,6,6,6,6,6,6,6)^T$ with error constants

$$c_{p+2} = \left[-\frac{9790337707657}{9790337707657}, -\frac{241}{223205220}, \frac{181}{685843200}, \frac{23}{102060}, -\frac{47}{60480}, -\frac{3449334093 79}{4048909834991 6160}, \frac{1231}{148803480}, -\frac{23}{9797760}, \frac{341}{612360}, -\frac{241}{72576}\right]^T$$

B. *Zero-stability*

A linear multistep method is said to be zero-stable if the roots $z_i,\ i = 1, 2, \ldots, k$ (grid and non-grid points) of the first characteristics polynomial defined by $\rho(z) = det(zA^0 - D)$ satisfies $|z_i| < 1$ and the root $|z| = 1$ having multiplicity not exceeding one (Lambert [20]). For our method,

$$\rho(z) = \left[z \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&1&0&0&0&0&\frac{13}{54} \\ 0&0&0&0&1&0&0&0&0&\frac{2}{3} \\ 0&0&0&0&1&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0&2 \\ 0&0&0&0&1&0&0&0&0&3 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \right]$$

$$= z^8(z-1)^2 = 0$$

Finding its determinant and solving the characteristic equation gives

$$\rho(z) = z^8(z-1)^2 = 0, \quad z = 0,0,0,0,0,0,0,0,1,1$$

Since, $|z_i| = |0,0,0,0,0,0,0,0,1,1| \leq 1$, the method is zero-stable.

C. Consistency

The method, (TOSB) is consistent since it has order $p = 6 \geq 1$

D. Convergence

According to Henrici [10], Eq. (6) converges if it is zero stable and consistent. Hence, the new method converges

IV. NUMERICAL RESULTS

In order to examine the validity of the new method (TOBS), the problems are solved. All computations are done using MAPLE 18.

Notations used in the result Tables are as follow:

KAMARUN = Error in Kamarun et al in prob 1

ADEGBORO = Error in Akinukawe & Akingbulu in prob2

ADENIRAN = Error in Adeniran & Edaogbogun in prob 3

ETOBS = Error in Three-step Optimized Block Scheme

where $\quad \text{Error}^{(i)} = \left|\Upsilon^{(i)}_{\text{exact}} - \Upsilon^{(i)}_{\text{approx.}}\right|$

Problem 1 [11]

$$\Upsilon'' = \Upsilon, \quad \Upsilon(0) = 1, \ \Upsilon'(0) = -10;$$

The exact solution given by

$$\Upsilon(x) = \exp(-10x)$$

Problem 2 [12]

$$\Upsilon'' = 100\Upsilon, \quad \Upsilon(0) = 1, \ \Upsilon'(0) = -10;$$

The exact solution is $\Upsilon(x) = \exp(-10x)$

Problem 3 [13]

$$\Upsilon'' + 1001\Upsilon' + 1000\Upsilon = 0, \ \Upsilon(0) = 1, \ \Upsilon'(0) = 1$$

The exact solution is given by $\Upsilon(x) = \exp(x)$.

Table I. Comparison of results and errors for Problem 1 [11]

x	Υ_{exact}	TOBS	ETOBS	KAMARUN *et al* [8]
0.01	1.10517091807564762480	1.10517091807564738530	2.395e-16	2.41140e-13
0.02	1.22140275816016983390	1.22140275815762476850	2.54507e-12	3.061329e-12
0.03	1.34985880757600310400	1.34985880758499218520	8.98908e-12	1.063993e-11
0.04	1.49182469764127031780	1.49182469768850932610	4.72390e-11	5.192646e-11
0.05	1.64872127070012814680	1.64872127078265503680	8.25269e-11	8.894574e-11
0.06	1.82211880039050897490	1.82211880052818871180	1.37680e-10	1.498492e-10
0.07	2.01375270747047652160	2.01375270770062814100	2.30151e-10	2.493450e-10
0.08	2.22554092849246760460	2.22554092881275800690	3.20290e-10	3.448744e-10
0.09	2.45960311115694966380	2.45960311159628439640	4.39335e-10	4.748948e-10
1.10	2.71828182845904523540	2.71828182907033670520	6.11292e-10	6.597363e-10

Table II. Comparison of results and errors for Problem 2 [12]

x	Y_{exact}	TOBS	ETOBS	Adegboro & Akingbulu [12]
0.1	0.90483741803595957316	0.90483741803595985544	2.8228e-16	1.064040e-09
0.2	0.8187307530779818586	0.8187307530759803745	2.00148413e-12	2.022018e-09
0.3	0.74081822068171786607	0.74081822068841048845	6.69262238e-12	2.118282e-09
0.4	0.67032004603563930074	0.67032004607134360642	3.570430568e-11	3.564361e-09
0.5	0.60653065971263342360	0.60653065977622359703	6.359017343e-11	4.188832e-09
0.6	0.54881163609402643263	0.54881163619407744289	1.0005101026e-10	4.776991e-09
0.7	0.49658530379140951470	0.49658530394392471553	1.5251520083e-10	4.821770e-09
0.8	0.44932896411722159143	0.44932896432262865705	2.0540706562e-10	5.805406e-09
0.9	0.40656965974059911188	0.40656966000683483830	2.6623572642e-10	6.272491e-09
1.0	0.36787944117144232160	0.36787944151228504750	3.408427259e-10	6.749911e-09

Table III Comparison of results and errors for Problem 3 [13]

x	Y_{exact}	TOBS	ETOBS	Adeniran & Eda.[13]
0.1	0.95122942450071400909	0.95122942450071402137	1.228e-17	3.5092e-13
0.2	0.90483741803595957316	0.90483741803595483319	4.73997e-15	1.0314e-11
0.3	0.86070797642505780723	0.86070797642510641061	4.860338e-14	1.9954e-11
0.4	0.81873075307798185867	0.81873075307802748248	4.562381e-14	2.7903e-11
0.5	0.77880078307140486825	0.77880078307168119917	2.7633092e-13	3.4194e-11
0.6	0.74081822068171786607	0.74081822068037219837	1.34566770e-12	3.9041e-11
0.7	0.70468808971871343435	0.70468808971745171328	1.26172107e-12	4.2656e-11
0.8	0.67032004603563930074	0.67032004602731776797	8.32153277e-12	4.5229e-11
0.9	0.63762815162177329314	0.63762815170930924637	8.7535953e-11	4.6927e-11
1.0	0.60653065971263342360	0.60653065978774870024	7.51152766e-11	4.7892e-11

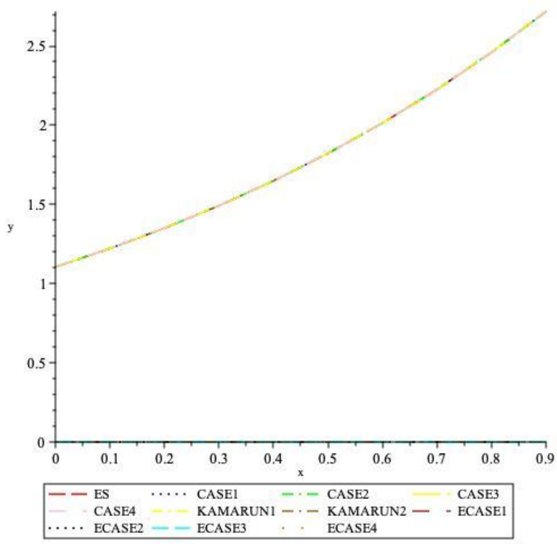

Fig. 1. Curves of Table I

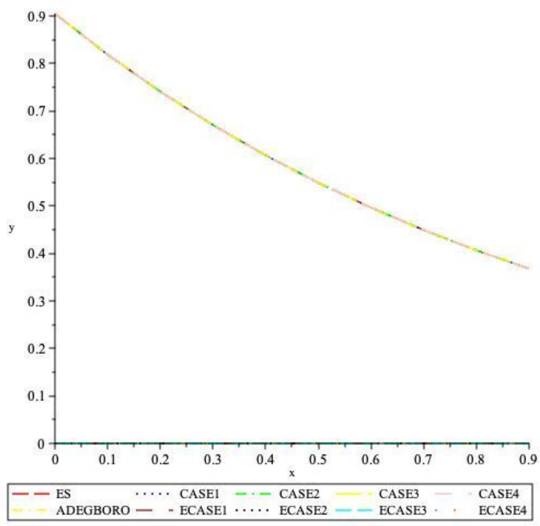

Fig. 2. Curves of Table II

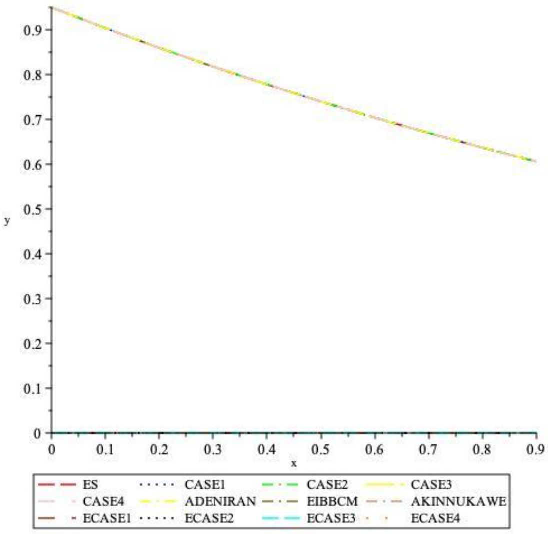

Fig. 3. Curves of Table III

V. DISCUSSION

Table I compared the exact solution, computed solution and errors of the new method and Kamarun *et al,* [11] when Problem 1 is solved. The efficiency of the new scheme over EIK is also shown in Fig. 1. Tables II and III, as well as Fig. 2 and Fig. 3, reveal the exact and computed solutions of the newly developed method, TOBS, for the solution of Problems 2 and 3. The new method performs better than method proposed by Kamarun *et al*, [11], and Adegboro & Akingbulu [12] in terms of accuracy. Similarly, the numerical results in Table III show that the new block method (TOBS) is a very effective and efficient scheme for solving initial value problems of fourth-order.

CONCLUSIONS

The This paper formulates and implements Obreckoff's type optimized hybrid scheme for solving second-order ordinary differential equation via interpolation and collocation methods. This method used the approximated power series as the basis function with its second derivatives as collocating equation. The analyses of the method were shown to be consistent, convergent and zero stable. The new scheme has proven to be an effective and efficient solution to second-order ODEs based on the numerical results shown in Tables I, II and III and in Figures 1 - 3.

REFERENCES

1. Akinnukawe, B.I. and Okunuga, S.A. (2024). One-step block scheme with optimal hybrid points for the numerical integration of second-order ordinary differential equations. Journal of the Nigerian Society of Physical Sciences, 6(2), 1827, DOI https://doi.org/10.46481/jnsps.2024.1827
2. Adeyeye, O. and Omar, Z. (2016). Maximal order block method for the solution of second order differential equations. International Journal of Applied Mathematics, 46 (3).
3. Shokiri, A. (2015). The symmetric two-step p-stable nonlinear predictor-corrector method for the numerical solution of second order initial value problems. Bulletin of the Iranian Mathematical Society 41 (1), 201–215. DOI https://bims.ims.ir.
4. James, A. A., Adesanya, A.O. and Sunday, J. (2015). Cntinuous block method for the solution of second order initial value problems of ordinary differential equations, International Journal of Pure and Applied Mathematics 83 (3), 405-416. DOI http://dx.doi.org/10.12732/ijpam.v83i3.3.
5. Ajinuhi, J.O., Mohammad, U., Enagi, A.I. and Jimoh, J.I. (2023). An optimized single-step block hybrid Nystrom method for solving second order initial value problems of Bratu-type. African Journal of Mathematics and Statistics Studies, 6(4), 18-27. DOI https://doi/10.52589/AJMSS- EDNC4FYA.
6. Hussain, K., Adeyeye, O. and Ahmad, N. (2023). Block method for second-order IVPs. Journal of the Nigerian Society of Physical Sciences, 5 (2), 1087, DOI https://doi.org/10.46481/jnsps.2023.1087.
7. Raymond, D., James, K. and Barde, W. (2024). Exponentially-fitted one-step four hybrid point method for solving stiff and oscillation problems, Asian Journal of Pure and Applied Mathematics, 6(4), 1-11.
8. Blest, D. (1990). Reduction of order as a neglected method for the solution of inhomogeneous linear ordinary differential equations. International Journal of Mathematical Education in Science and Technology, 21(3), 393-401, DOI https://doi.org10.1080/0020739900210307.
9. AbdulAzeez, K.J. (2024). Two-step hybrid block method for the solution of second order initial value problem of ordinary differential equations. Earthline Journal of Mathematical Society, 14(5), 1047-1065. DOI https://doi.org/10.34198/ejms.14524.10471065.
10. Henrici, P. (2008). Discrete variable methods in ODEs. First Edition, John Wiley and Sons, Ltd.
11. Kamarun, H.Z., Adeyeye, O. and Zurni, O. (2023). Three step block methods with different off-step points interval for solving second order initial value problems. Journal of Theoretical and Applied Information Technology, 101(16), 6310-6320.
12. Adegboro, J.O. and Akingbulu, O.B. (2023). A class of continuous linear multistep method for solving second order ordinary differential equations. International Journal of Innovation Science and Research Technology, 8 (28), 2886-2894.
13. Adeniran, A.O. and Edaogbogun, K. (2021). Half step numerical method for the solution of second order

initial value problems. Academic Journal of Applied Mathematical Sciences, 7(2), 1-18, DOI https://doi.org/10.32861/ajams.72.77.81.

14. Jator, S. and Li, J. (2009). A self-starting linear multistep method for a direct solution of the general second-order initial value problem. International Journal of Computer Mathematics, 86(5), 825-836, DOI https://doi.org/10.1080/00207160701708250.
15. Joao, M. S. A., Paulo, F., Jorge, F. V and Luis, M.S. (2007). Outstanding issues in model order reduction. Mathematics in Industry 11, 139-152, DOI https://doi.org/10.1007/978-3-540-71980-913.
16. Fatunla, S.O. (1988). Numerical methods for initial value problems in ordinary differential equations. New York Academic Inc.
17. Areo, E. A., Gbenro, S., Olabade, B. and Momoh, A. (2024). One-step three-parameter optimized hybrid block method for solving first order initial value problems of ordinary differential equations. Journal of Numerical Analysis and Modeling, 5(1), 41-51, DOI https://doi.org/10.48185/jnam.v5il.970.
18. Lamiaa H. A. and Ahmed, A. M. F. (2022). Order reduction method and its application to singular perturbed delay differential equations. Mathematical Statistician and Engineering Applications, 71(4), 5110-5124, DOI https://doi.org/10.17762/msea.v7li4.1100.
19. Yakubu D. G. and Markus, S. (2016). The efficiency of second derivative multistep methods for the numerical integration of stiff systems., Journal of the Nigerian Mathematics Society, 35, 107-108.
20. Lambert, J.D. (1973). Computational mathematics in ordinary differential equations. John Wiley and Sons, Inc.