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Optimized Hybrid Two-Step Four Parameter Block Method for Solving First-Order Ordinary Differential Equations

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ABSTRACT: Using power series and exponential functions as the basis function, this work proposes a two-step, four-parameter optimized hybrid block technique for solving first-order ODEs. This method approximates and generates several continuous schemes. The suggested approach incorporates four free parameters to improve accuracy and flexibility. The method is appropriate for both stiff and non-stiff issues since optimization techniques were used to enhance it and reduce the local truncation error. The optimized continuous method gave rise to the discrete schemes, which were then utilized to concurrently solve initial value problems in block mode. The method's convergence, zero-stability, and consistency are confirmed by a thorough numerical study. The outcomes demonstrate the adaptability and efficiency of the suggested approach in solving a variety of ODE issues, providing a viable instrument for scientific and practical uses.

KEYWORD: optimized hybrid method, Free-Parameter, exponentially fitted, local truncation error

1. INTRODUCTION

In the scientific and technical fields, differential equations are a natural part of the mathematical formulation of physical processes. It is frequently difficult to find analytical solutions to the majority of differential equations. In this paper, it will be considered the numerical solution of the first order initial value problem of ordinary differential equation of the form

$$w' = \gamma(x, w), \; w(x_0) = w_0, \; x \in (m, n) \tag{1.1}$$

To get an approximate solution of (1), numerical techniques had to be used. The construction of continuous linear multistep methods for the direct solution of initial value problems in ordinary differential equations has been extensively studied as can be seen in some scholars work most especially in the work of Akinfenwa et al., 2020, Akinfenwa et al., 2013, Areo et al., 2024.

In their 2019 study, Ramos suggested a two-step process that included optimizing the LTEs to choose two intermediate spots. Although the approach was reformed as an R-K method, it required more computing power to be accomplish. For the purpose of optimizing first-order initial value problems (IVPs), Kashkari and Syam (2019) introduced a novel optimized one-step hybrid block technique. Gbenro (2025) proposed an optimized hybrid block methods with high efficiency for the solution of first-order ordinary differential equations. In order to optimize the LTEs of the fundamental equations driving the block's behavior, the procedure involved carefully choosing three hybrid points.

The novelty of this paper is the introduction of a free-parameter among the two-step four-parameter optimized hybrid block method which is capable of solving efficiently the IVP in (1).

The paper is organized as follows; the derivation of the method is based on the free-parameter chosen and the values assigned to it before and after optimizing the LTE. The third section present the analysis which is the order, error constant, stability region. The fourth section provide the numerical efficiency of the method as compare to existing work in the literature and the conclusion follows.

Definition 1: Optimization is a mathematical formulation that allows to minimize or maximize the particular objective function subjected to a constraints on its variables.

Definition 2: Hybrid Block algorithm are a class of numerical methods that blend linear multistep approaches with interpolation using basis functions.

Definition 2: A stiff equation is a differential equations that are characterized as those whose exact solutions has a term of the form e^{-ct} where c is large positive constant.

2. DERIVATION OF THE METHOD

To derive this, we combine the power series and exponential function as the basis function which is given of the form

$$w(x) = \left(\sum_{j=0}^{f} b_j x^j + \sum_{j=1}^{g} \psi_j e^{x^j} \right) \tag{2}$$

b_j and ψ_j are real constants to be determine. The OHBTM shall be a system at the grid points r, s, u, v which shall be implemented in block.

Differentiating (2) once gives

$$w'(x)=p'(x)=\left(\sum_{j=0}^{3}\rho_j\, j!\, x^{j-1}+\sum_{j=1}^{4} jx\ell^{x^{j-1}}\right) \tag{3}$$

Evaluating (2) at the points $x_{n+q}, q=0$ and collocating (3) at all points $x_{n+\psi}=x_{n+\psi h}, \psi=\{0,r,s,1,u,v,2\}$ leads to a system of eight algebraic equations for the unknowns $\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7$ which when expressed in matrix form yields

$$\begin{bmatrix}
2 & 2x_n & \frac{3}{2}x_n^2 & \frac{7}{6}x_n^3 & \frac{25}{24}x_n^4 & \frac{121}{120}x_n^5 & \frac{721}{720}x_n^6 & \frac{1}{5040}x_n^2 \\
0 & 2 & 3x_n & \frac{7}{2}x_n^2 & \frac{25}{6}x_n^3 & \frac{121}{24}x_n^4 & \frac{721}{120}x_n^5 & \frac{1}{720}x_n^6 \\
0 & 2 & 3(x_n+rh) & \frac{7}{2}(x_n+rh)^2 & \frac{7}{2}(x_n+rh)^3 & \frac{7}{2}(x_n+rh)^4 & \frac{7}{2}(x_n+rh)^5 & \frac{7}{2}(x_n+rh)^6 \\
0 & 2 & 3(x_n+sh) & \frac{7}{2}(x_n+sh)^2 & \frac{7}{2}(x_n+sh)^3 & \frac{7}{2}(x_n+sh)^4 & \frac{7}{2}(x_n+sh)^5 & \frac{7}{2}(x_n+sh)^6 \\
0 & 2 & 3(x_n+h) & \frac{7}{2}(x_n+h)^2 & \frac{7}{2}(x_n+h)^3 & \frac{7}{2}(x_n+h)^4 & \frac{7}{2}(x_n+h)^5 & \frac{7}{2}(x_n+h)^6 \\
0 & 2 & 3(x_n+uh) & \frac{7}{2}(x_n+uh)^2 & \frac{7}{2}(x_n+uh)^3 & \frac{7}{2}(x_n+uh)^4 & \frac{7}{2}(x_n+uh)^5 & \frac{7}{2}(x_n+uh)^6 \\
0 & 2 & 3(x_n+vh) & \frac{7}{2}(x_n+vh)^2 & \frac{7}{2}(x_n+vh)^3 & \frac{7}{2}(x_n+vh)^4 & \frac{7}{2}(x_n+vh)^5 & \frac{7}{2}(x_n+vh)^6 \\
0 & 2 & 3(x_n+2h) & \frac{7}{2}(x_n+2h)^2 & \frac{7}{2}(x_n+2h)^3 & \frac{7}{2}(x_n+2h)^4 & \frac{7}{2}(x_n+2h)^5 & \frac{7}{2}(x_n+2h)^6
\end{bmatrix}
\begin{bmatrix}\lambda_0\\ \lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_5\\ \lambda_6\\ \lambda_7\end{bmatrix}
=
\begin{bmatrix}\tau_n\\ \gamma_0\\ \gamma_{n+r}\\ \gamma_{n+s}\\ \gamma_{n+1}\\ \gamma_{n+u}\\ \gamma_{n+v}\\ \gamma_{n+2}\end{bmatrix} \tag{4}$$

Solving equation (4) by Gaussian Elimination method to get the coefficients $\lambda_j^{;s}$, $j=0,r,s,1,u,v,2$ and substituting back into equation (3) yield the optimized continuous two-step hybrid method of the form

$$p(x)=\lambda_j\tau_{n+j}+h\left[\sum_{i=r,s,u,v}\lambda_i\gamma_{n+i}+\sum_{j=0}^{2}\lambda_i\gamma_{n+i}\right] \tag{5}$$

Evaluating equation (5) at the points $\lambda=\lambda_j,\lambda_r,\lambda_s,\lambda_1,\lambda_u,\lambda_v,\lambda_2,\ j=0$ yield the following(6)

$$\lambda_r=-\frac{1}{420}\frac{28s+28u+28v-49su-49sv-49uv+105suv-18}{r(r-1)(r-2)(r-v)(r-u)(r-s)} \tag{7}$$

$$\lambda_s=\frac{1}{420}\frac{28r+28u+28u+28v-49ru-49rv-49rv-49uv+105ruv-18}{s(s-1)(s-2)(s-v)(s-u)(r-s)} \tag{8}$$

$$\lambda_1=\frac{1}{420}\frac{-98r-98s-98u-98v+126rs+126ru+126rv+126su+126sv+126uv-175rsu-175rsv-175ruv-175s}{(v-1)(u-1)(s-1)(r-1)} \tag{9}$$

$$\lambda_u=-\frac{1}{420}\frac{28r+28s+28v-49rs-49rv-49sv+105rsv-18}{u(u-1)(u-2)(u-v)(s-u)(r-u)} \tag{10}$$

$$\lambda_v=\frac{1}{420}\frac{28r+28s+28u-49rs-49ru-49su+105rsu-18}{v(v-1)(v-2)(u-v)(s-v)(r-v)} \tag{11}$$

$$\lambda_2=-\frac{1}{840}\frac{-14r-14s-14u-14v+21rs+21ru+21rv+21su+21sv+21uv-35rsu-35rsv-35ruv-35suv+70rsuv+10}{(v-2)(u-2)(s-2)(r-2)} \tag{12}$$

The principal term of the local truncation error in λ_{n+1} is computed and used to calculate the appropriate values for the unknown parameters.

$$L[w(\lambda_{n+1});h]=\frac{1}{2116800}\begin{pmatrix}-36r-36s-36v-36u+56rs+56ru+56rv+56su+56sv\\ +56uv-98rsu-98rsv-98ruv-98suv+210rsuv+25\end{pmatrix}+0(h^9) \tag{13}$$

Setting the principal term of the local truncation error in (13) to zero yield;

$$\frac{-36r-36s-36v-36u+56rs+56ru+56rv+56su+56sv+56uv-98rsu-98rsv-98ruv-98suv+210rsuv+25}{2116800}=0 \tag{14}$$

The variable r is optimized in (14) while $s,u, and\ v$ are treated as free parameters by assigning the values as $s=\frac{2}{3}$, $u=\frac{4}{3}$, $v=\frac{5}{3}$ yielding;

$$r=\frac{17}{72} \tag{15}$$

By substituting the optimize value of $r=\frac{17}{72}$ and that of s,u,v into (4) we obtained the discrete scheme

	τ_n	γ_n	$\gamma_{n+\frac{17}{72}}$	$\gamma_{n+\frac{2}{3}}$	γ_{n+1}	$\gamma_{n+\frac{4}{3}}$	$\gamma_{n+\frac{5}{3}}$	γ_{n+2}
$\tau_{n+\frac{17}{72}}$	1	$\frac{2938281888\ 19}{3467372396\ 544}$	$\frac{1112098521\ 7}{5920136191\ 2}$	$-\frac{1457889604\ 525}{1791475738\ 214}$	$\frac{1820655260\ 777}{4303818522\ 624}$	$-\frac{6623344870\ 549}{4421791218\ 4832}$	$-\frac{1041302926\ 457}{7440403267\ 5840}$	$-\frac{4289411486\ 053}{2201781471\ 805440}$
$\tau_{n+\frac{2}{3}}$	1	$\frac{4261}{64260}$	$\frac{1596771532\ 8}{4193429802\ 1}$	$\frac{33241}{117180}$	$-\frac{976}{10395}$	$\frac{3883}{99540}$	$-\frac{1024}{97335}$	$\frac{209}{160020}$
τ_{n+1}	1	$\frac{1573}{22848}$	$\frac{1547934105\ 6}{4193429802\ 1}$	$\frac{30897}{69440}$	$\frac{551}{4620}$	$-\frac{297}{176960}$	$-\frac{9}{14420}$	$\frac{97}{853440}$
$\tau_{n+\frac{4}{3}}$	1	$\frac{3238}{48195}$	$\frac{7881936076\ 8}{2096714901\ 05}$	$\frac{12088}{29295}$	$\frac{10048}{31185}$	$\frac{4234}{24885}$	$-\frac{320}{19457}$	$\frac{608}{360045}$
$\tau_{n+\frac{5}{3}}$	1	$\frac{14285}{205632}$	$\frac{1539440540\ 0}{4193429802\ 1}$	$\frac{167375}{374976}$	$\frac{1975}{8316}$	$\frac{135875}{318528}$	$\frac{9565}{77851}$	$-\frac{1625}{512064}$
τ_{n+2}		$\frac{89}{1428}$	$\frac{8255648563\ 2}{2096714901\ 05}$	$\frac{1539}{4340}$	$\frac{496}{1155}$	$\frac{1971}{11060}$	$\frac{1728}{3605}$	$\frac{5459}{53340}$

(16)

3 ANALYSIS OF THE NEW BLOCK METHOD

3.1 Order and error constant

$$R_1X_n = R_0X_{n-1} + h\left(T_0F_{n-1} + T_1F_n\right)$$

Where R_0, R_1, T_0 and T_1 are 6x6 matrices given by

$$R_1=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix};\ R_0=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix};\ T_0=\begin{bmatrix}0&0&0&0&0&\frac{2938281888\ 19}{3467372396\ 544}\\0&0&0&0&0&\frac{4261}{64260}\\0&0&0&0&0&\frac{1573}{22848}\\0&0&0&0&0&\frac{3238}{48195}\\0&0&0&0&0&\frac{14285}{205632}\\0&0&0&0&0&\frac{89}{1428}\end{bmatrix}$$

$$T_1 = \begin{bmatrix} \frac{11120985217}{59201361912} & -\frac{14578896045253}{179147573821440} & \frac{1820655260777}{238381852256240} & -\frac{6623344870549}{152179121848320} & \frac{1041302926457}{744040322675840} & -\frac{42894114486053}{2201781471805440} \\ \frac{15967715328}{41934298021} & \frac{33241}{117180} & -\frac{976}{10395} & \frac{3883}{99540} & -\frac{1024}{97335} & \frac{209}{160020} \\ \frac{15479341056}{41934298021} & \frac{30897}{69440} & \frac{551}{4620} & -\frac{297}{176960} & -\frac{9}{14420} & \frac{97}{853440} \\ \frac{78819360768}{209671490105} & \frac{12088}{29295} & \frac{10048}{31185} & \frac{4234}{24885} & -\frac{320}{19467} & \frac{608}{360045} \\ \frac{15394406400}{41934298021} & \frac{167375}{374976} & \frac{1975}{8316} & \frac{135875}{318528} & \frac{9565}{77868} & -\frac{1625}{512064} \\ \frac{82556485632}{209671490105} & \frac{1539}{4340} & \frac{496}{1155} & \frac{1971}{11060} & \frac{1728}{3605} & \frac{5459}{53340} \end{bmatrix}$$

$\tau_m = \left(\tau_{m+r}, \tau_{m+s}, \tau_{m+1}, \tau_{m+u}, \tau_{m+v}, \tau_{m+2}\right)^T$,

$\tau_{m-1} = \left(\tau_{m-1+r}, \tau_{m-1+s}, \tau_m, \tau_{m-1+u}, \tau_{m-1+v}, \tau_{m+1}\right)^T$,

$\gamma_m = \left(\gamma_{m+r}, \gamma_{m+s}, \gamma_{m+1}, \gamma_{m+u}, \gamma_{m+v}, \gamma_{m+2}\right)^T$,

$\gamma_{m-1} = \left(\gamma_{m-1+r}, \gamma_{m-1+s}, \gamma_m, \gamma_{m-1+u}, \gamma_{m-1+v}, \gamma_{m+1}\right)^T$,

The 2SOHBM is of order $p = \left(7,7,8,7,7,7\right)^T$

The order of the new method is seventh with the error constant

$$C_{p+1} = \left[\frac{4456957284725}{5823854155193647104}, -\frac{43}{148803480}, -\frac{23}{1343692800}, -\frac{1}{3720087}, \frac{25}{119042784}, -\frac{1}{612360}\right]^T$$

3.2 zero-stability

A linear multistep method is said to be zero-stable if the roots $z_i, i = 1,2,...,k$ (grid and non-grid points) of the first characteristics polynomial defined by $\rho(z) = \det\left(zA^0 - D\right)$ satisfies $|z_i| < 1$ and the root $|z| = 1$ having multiplicity not exceeding one. For our method,

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = z^5\left(z-1\right)$$

$$\rho(z) = (z-1)z^5 = 0,\ z = 0,0,0,0,0,1$$

Since $|z_i| = |0,0,0,0,0,1| \leq 1$ the method is zero-stable.

3.3 Consistency

This method, (2SOHM) is consistent since it has order p =7 ≥ 1.

3.4 Convergence

Zero-stable and consistency are sufficient conditions for a linear multistep method to be convergent according to (Henrici, 1962)

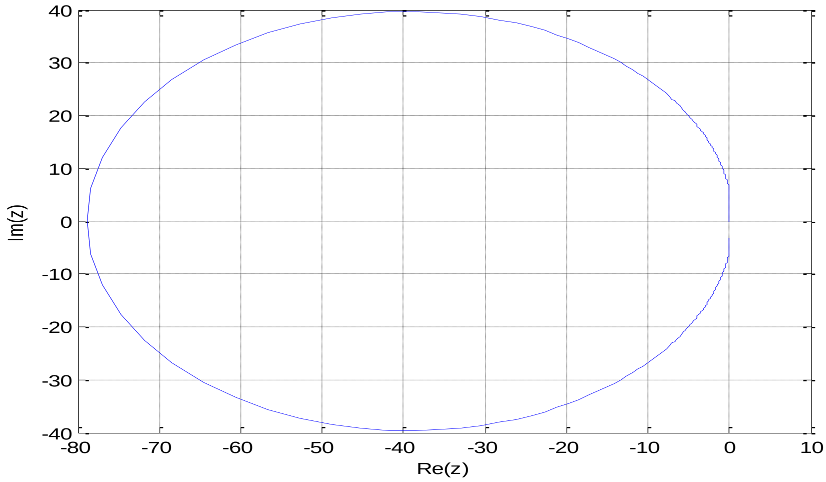
3.4 Region of Absolute Stability

The stability polynomial for (16) gives

$$h^6\left(\tfrac{17}{122472}w^6 - \tfrac{127}{612360}w^5\right) - h^5\left(\tfrac{139}{58320}w^6 + \tfrac{59}{19440}w^5\right) + h^4\left(\tfrac{1247}{58320}w^6 - \tfrac{1457}{58320}w^5\right) - h^3\left(\tfrac{473}{3888}w^6 + \tfrac{523}{3888}w^5\right)$$

$$-h^2\left(\tfrac{77}{162}w^5 - \tfrac{145}{324}w^6\right) - h\left(\tfrac{71}{72}w^6 + \tfrac{73}{72}w^5\right) + w^6 - w^5$$

and its region is shown in Figure 1



4. NUMERICAL RESULTS

The following problems were solved in order to examine the validity of the new method. All computations were done using Maple 18.

Problem 1

Consider the SIR model problem from Mufutau et al. (2024) which arises in mathematical Biology. The SIR model describes infectious disease dynamics over time. It involves the following three connected differential equations representing susceptible $S(t)$ infected $I(t)$ and recovered $R(t)$ individuals:

$$\frac{dS}{dt} = \Delta(1-S) - \alpha IS, \tag{4.1}$$

$$\frac{dI}{dt} = -\Delta I - \theta I + \alpha IS \tag{4.2}$$

$$\frac{dR}{dt} = \Delta R + \theta I \tag{4.3}$$

Where Δ, θ and α are positive parameters. Additionally, we introduce the function $v(t)$ which is defined as the sum of the susceptible, infected and recovered individuals at time $v(t)$. this function is given by

$$v(t) = S(t) + I(t) + R(t) \tag{4.4}$$

By adding equation (4.1) – (4.3), we obtain the following expression for the derivative of $v(t)$

For a particular closed population assuming an initial condition of $v(0) = 0.5$ and $\Delta = 0.5$, it can be written as (Mufutau et al.,2024):

$$v'(t) = 0.5(1 - v(t)), \quad v(0 = 0.5) \qquad t \in [0,1]$$

$$Exact\ \ solution: v(t) = 1 - 0.5\exp(-0.5t)$$

Problem 2

Consider the given equation as in (Adoghe et al.,2024):

$$y' = -y$$

$$h = \frac{1}{10}, \quad y(0) = 1,$$

$$Exact\ \ solution: y_1(x) = e^{-x},$$

Problem 3

Consider the following stiff system of equation given by Umar et al.,(2022)

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 198 & 199 \\ -398 & -399 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad h = \frac{1}{10}, \quad y_1(0) = 1, y_2(0) = 1$$

Table1: Comparison of results and errors for problem 1

x	Exact Solution	Computed Solution in our Method	Error in our method	Error in Mufutau et al, (2024)
0.1	0.52438528774964299546	0.52438528774964299533	1.30e-19	4.000e-20
0.2	0.54758129098202021342	0.54758129098202018321	3.021e-17	4.547e-16
0.3	0.56964601178747109638	0.56964601178747106754	2.884e-17	4.3254e-16
0.4	0.59063462346100907066	0.59063462346100901601	5.465e-17	8.2288e-16
0.5	0.61059960846429756588	0.61059960846429751379	5.209e-17	7.8277e-16
0.6	0.62959088965914106696	0.62959088965914099281	7.415e-17	1.11688e-16
0.7	0.64765595514064328282	0.64765595514064321219	7.063e-17	1.06241e-16
0.8	0.66483997698218034963	0.66483997698218026017	8.946e-17	1.34748e-15
0.9	0.68118592418911335343	0.68118592418911326826	8.517e-17	1.28174e-15
1.0	0.69673467014368328820	0.69673467014368318704	1.0116e-16	1.52406e-15

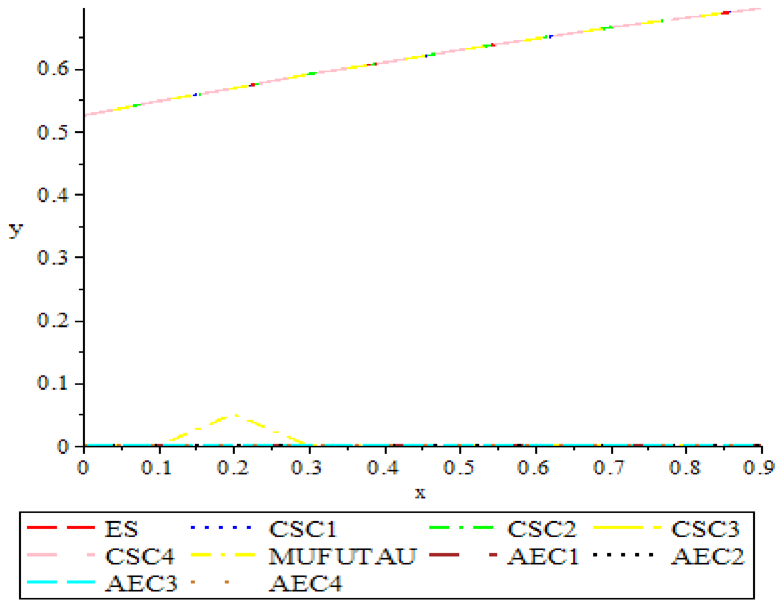


Figure 1: Graphical Illustration of the Error in the new methods and error in Mufutau (2024)

Table2: Comparison of results and errors for problem 2

x	Exact Solution	Computed Solution in our Method	Error in our method	Error in Adoghe et al, (2024)
0.1	0.90483741803595957316	0.90483741803595968526	1.121e-16	1.4969669e-13
0.2	0.81873075307798185867	0.81873075307799654478	1.468611e-14	2.7090233e-13

0.3	0.74081822068171786607	0.74081822068173124638	1.338031e-14	3.6768384e-13
0.4	0.67032004603563930074	0.67032004603566334868	2.404794e-14	4.4359214e-13
0.5	0.60653065971263342360	0.60653065971265525822	2.183462e-14	5.0172346e-13
0.6	0.54881163609402643263	0.54881163609405596581	2.953318e-14	5.4477378e-13
0.7	0.49658530379140951470	0.49658530379143629895	2.678425e-14	5.7508699e-13
0.8	0.44932896411722159143	0.44932896411725383106	3.223963e-14	5.9469740e-13
0.9	0.40656965974059911188	0.40656965974062833387	2.922199e-14	6.0536752e-13
1.0	0.36787944117144232160	0.36787944117147531607	3.299447e-14	6.0862131e-13

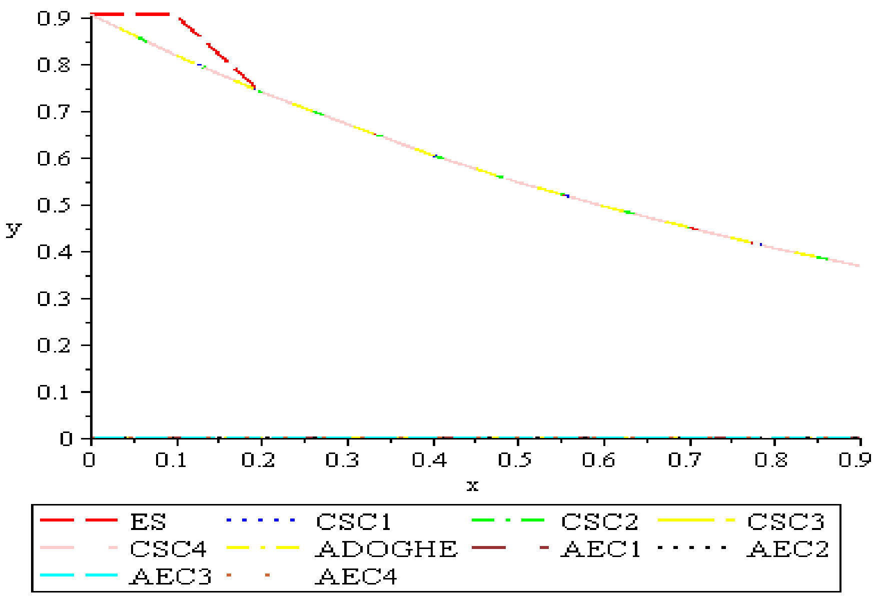


Figure 4. 1: Graphical Illustration of the Error in the new method and Error in Adoghe et al (2024)

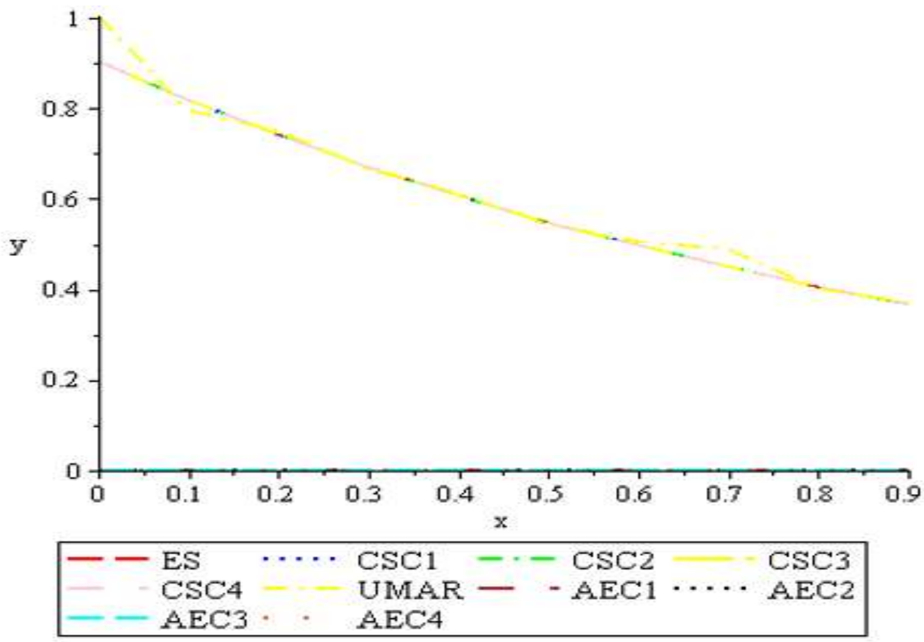
Table3: Comparison of results and errors for problem 3

x	Exact Solution y1	Computed Solution in our Method	Error in our method	Error in Umar et al, (2022)
0.1	0.90483741803595957316	0.90483741803595968487	1.1171e-16	1.004e-00
0.2	0.81873075307798185867	0.81873075307799654701	1.468834e-14	7.956e-01
0.3	0.74081822068171786607	0.74081822068173124873	1.338266e-14	7.499e-01
0.4	0.67032004603563930074	0.67032004603566335241	2.405167e-14	6.647e-01
0.5	0.60653065971263342360	0.60653065971265526223	2.183863e-14	6.113e-01
0.6	0.54881163609402643263	0.54881163609405597058	2.953795e-14	5.431e-01
0.7	0.49658530379140951470	0.49658530379143630400	2.67893e-14	5.034e-01
0.8	0.44932896411722159143	0.44932896411725383656	3.224513e-14	4.896e-01

0.9	0.40656965974059911188	0.40656965974062833945	2.922757e-14	4.025e-01
1.0	0.36787944117144232160	0.36787944117147532183	3.300023e-14	3.688e-01

Table3: Comparison of results and errors for problem 3

x	Exact Solution y2	Computed Solution in our Method	Error in our method	Error in Umar et al, (2022)
0.1	-0.90483741803595957316	-0.90483741803595968480	1.1164e-16	1.011e-00
0.2	-0.81873075307798185867	-0.81873075307799654640	1.468773e-14	7.933e-01
0.3	-0.74081822068171786607	-0.74081822068173124877	1.33827e-14	7.511e-01
0.4	-0.67032004603563930074	-0.67032004603566335133	2.405059e-14	6.639e-01
0.5	-0.60653065971263342360	-0.60653065971265526226	2.183866e-14	6.122e-01
0.6	-0.54881163609402643263	-0.54881163609405596956	2.953693e-14	5.420e-01
0.7	-0.49658530379140951470	-0.49658530379143630403	2.678933e-14	5.045e-01
0.8	-0.44932896411722159143	-0.44932896411725383572	3.224429e-14	5.094e-01
0.9	-0.40656965974059911188	-0.40656965974062833946	2.922758e-14	4.001e-01
1.0	-0.36787944117144232160	-0.36787944117147532110	3.29995e-14	3.694e-01



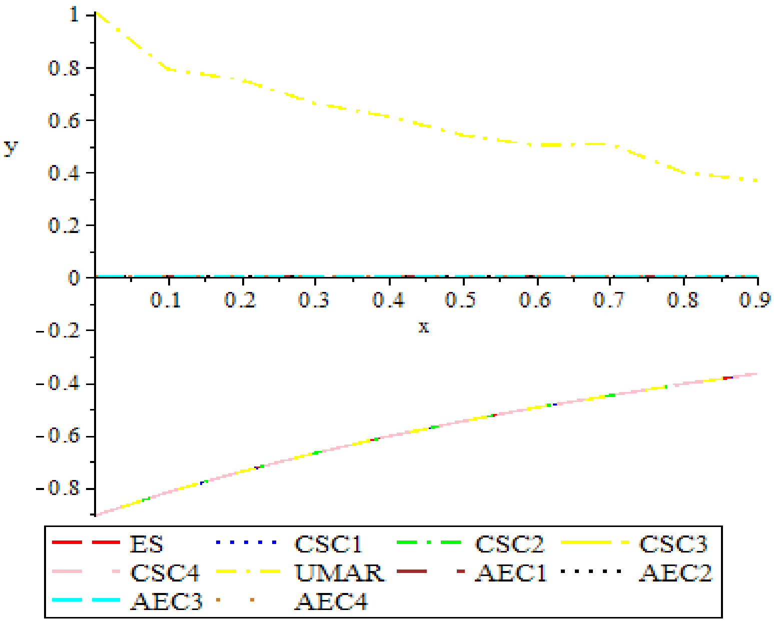


Figure 4. 2: Graphical Illustration of the Error in the new method and error in Umar (2022)

4 DISCUSSION AND CONCLUSION

We derived an A-stable two step optimized hybrid block method for the solution of first-order ordinary differential equations using multistep collocation approach. The method developed was also found to be zero-stable, consistent and convergent. The results obtained in tables 1, 2 and 3 from the three examples, Mufutau et al. (2024), Adoghe et al.,2024 and Umar et al.,(2022) shows that the new method competes favorably and provide better approximation than the existing methods. The order is higher than the existing work by one due to the optimized form of it.

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