Engineering and Technology Journal e-ISSN: 2456-3358

Volume 10 Issue 05 May-2025, Page No.- 4960-4970

DOI: 10.47191/etj/v10i05.24, I.F. – 8.482

Development of an Efficient Optimized Hybrid One Step Methods for Solving Fourth-Order Initial Value Problems (IVPs)

Raymond D.[1], Pantuvo T.P.[2], Kida M.[3], Ebenezer O.S.[4]

[1,2]Department of Mathematics, Federal University Wukari, Nigeria

[3]Department of Mathematics and Computer Science, Bornu State University, Maiduguri, Nigeria

[4]Department of Mathematics, Federal College of Education, Obudu, Nigeria

ABSTRACT: This paper examines the development of a one-step optimized block schemes for solving-fourth order initial value problems (IVP) of ordinary differential equations. Interpolation and collocation techniques are considered using approximated power series as the basis function and its fourth derivative as the collocating equation. Two hybrid points ω and δ are chosen to achieve optimized errors. The method's properties are analyzed and it is zero stable, consistent and convergent. The application of the method to some fourth-order initial value problems shows the derived method performed better in terms of accuracy and effectiveness; outperforming some existing methods when compared.

KEYWORDS: Collocation techniques, Optimized hybrid points, Initial value problems, Implicit hybrid method.

1. INTRODUCTION

This paper considers an effective solution to the initial value problems of the fourth-order differential equation of the form

$$\varphi^{(4)} = f\left(t, \varphi, \varphi', \varphi'', \varphi'''\right), \quad (1)$$

subject to the initial conditions

$$\varphi(t_0) = a_0, \varphi'(t_0) = a_1, \varphi''(t_0) = a_2, \varphi'''(t_0) = a_3$$

where $t \in [a, b]$ and f is continuous in $[a, b]$.

Such problems in (1) above frequently arise in physical sciences and engineering fields, e.g. structural mechanics, fluid dynamics, control theory, waves and vibration analysis [11]. Finding suitable solutions to fourth-order IVPs is usually challenging necessitating the need for numerical methods to provide suitable approximate solutions to such [9], [11]. In the past, use of reduction method to solve fourth-order ordinary differential equations was popular among researchers [11], [8]. This approach involves reducing the differentials to a system of first order ODEs, then applying appropriate numerical method to the first order to solve the system of equation obtained. This approach has resulted in significant computational challenges and inefficient utilization of computer resources, hence, affecting the accuracy of the method in terms of error. The method has been extensively analyzed in [1], [2} and [12].

Other schemes developed by scholars to handle the solution of fourth-order IVPs include the predictor-corrector method [3] and Taylors series [7] which have similar disadvantages as the reduction method. Some researchers applied direct methods to solve (1) which include Akinnukawe *et al* [9], Kuboye *et al* [11], Akinnukawe & Odekunle [10], Raymond *et al* [8], Awoyemi [7], Areo & Omole [1], Kuboye & Omar [4], Awoyemi [5], Familua [6] and lots more; but the accuracy of their methods can be in terms of error improved upon.

The essence of this study is to develop and implement a new scheme for the effective solution of (1) using optimized hybrid points ω and δ such that $0 < \omega < \delta < 1$ with the aim to produce an approximation with reduced error compared with existing methods.

2. DERIVATION OF THE ONE-STEP OPTIMIZED HYBRID METHODS

Let the approximate solution to Eq. (1) be a power series of the form

$$\varphi(t) = p(t) \approxeq \sum_{j=0}^{k} \rho_j t^j \quad (2)$$

Where $\rho \in \mathbb{R}$ are real unknown coefficients that will be determined.

The fourth derivative of Eq. (2) gives

$$\varphi^{(4)}(t) = p^{(4)}(t) = \sum_{j=4}^{k-1} \rho_j j\,(j-1)(j-2)(j-3)\; t^{j-4} = \kappa(t,\varphi,\varphi',\varphi'',\varphi'''), \tag{3}$$

$$j = 0, \ldots, 4.$$

Interpolating the first, second and third derivatives of equation (3) at t_n and collocating the fourth derivative of (2) at all points $t_{n+m} = t_n + mh$, $m = \{0, \omega, \delta, 1\}$, where ω and δ are the two hybrid points. In this way we obtain the system of nonlinear equations of the form

$$V = MD \tag{4}$$

where

$$M = \begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_N \end{bmatrix}, \quad V = \begin{bmatrix} \Phi_n \\ \Phi'_n \\ \Phi''_n \\ \Phi'''_n \\ K_n \\ K_{n+\omega} \\ K_{n+\delta} \\ K_{n+N} \end{bmatrix} \quad D = \begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 \\ 0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 \\ 0 & 0 & 0 & 0 & 24 & 120t_n & 360t_n^2 & 840t_n^3 \\ 0 & 0 & 0 & 0 & 24 & 120(t_n+\omega h) & 360(t_n+\omega h)^2 & 840(t_n+\omega h)^3 \\ 0 & 0 & 0 & 0 & 24 & 120(t_n+\delta h) & 360(t_n+\delta h)^2 & 840(t_n+\delta h)^3 \\ 0 & 0 & 0 & 0 & 24 & 120(t_n+h) & 360(t_n+h)^2 & 840(t_n+h)^3 \end{bmatrix}$$

Using Gaussian elimination method, to solve (4) gives the coefficient ρ_i's for i=0(1)7.
The obtained values of these coefficients, are then substituted into (2) to give a continuous implicit hybrid method of the form;

$$(t_n + Nh) = p(t) = \rho_0\varphi_n + h\rho_1\varphi'_n + h^2\rho_2\varphi''_n + \rho_3\varphi'''_n + h^4\left[\rho_4\kappa_n + \rho_5\kappa_{n+\omega} + \rho_6\kappa_{n+\delta} + \rho_7\kappa_{n+1}\right] \tag{5}$$

where

$$\rho_0 = 1, \rho_1 = N, \rho_2 = \frac{1}{2}N^2, \rho_3 = \frac{1}{6}N^3, \rho_4 = \frac{1}{1260}\frac{N^4(-7N\delta - 7N\omega + 42\delta\omega + 2N)}{\delta\omega}$$

$$\rho_5 = -\frac{1}{1260}\frac{N^6(2N - 7\delta)}{\omega(\omega - 1)(\delta - \omega)}, \; \rho_6 = \frac{1}{1260}\frac{N^6(2N - 7\omega)}{\delta(N - \delta)(\delta - \omega)} \tag{6}$$

By substituting N=1 in (5) and its derivatives (up to third derivative), we obtain a set of multistep formulas to approximate the solutions of (1) at the points $t_{n+1}, t'_{n+1}, t''_{n+1}$ and t'''_{n+1} respectively as

$$t_{n+1} = \varphi_n + h\varphi'_n + \frac{1}{2}h^2\varphi''_n + \frac{1}{6}h^3\varphi'''_n + h^4\left[\left(\frac{1}{1260}\frac{(-7\delta - 714\omega + 42\delta\omega + 2)}{\delta\omega}\right)\kappa_n\right] \tag{7}$$

$$+ h^4\left[-\left(\frac{1}{1260}\frac{(7\delta - 2)}{\omega(\omega - 1)(\delta - \omega)}\right)\kappa_{n+\omega} + \left(\frac{1}{1260}\frac{(7\omega - 2)}{\delta(\delta - 1)(\delta - \omega)}\right)\kappa_{n+\delta}\right]$$

$+$

$$ht'_{n+1}=h\varphi'_n+h^2\varphi''_n+\frac{1}{2}h^3\varphi'''_n+h^4\left[\left(-\frac{1}{120}\frac{(3\delta+\omega-15\delta\omega-1)}{\delta\omega}\right)\kappa_n\right]$$
$$-h^4\left[\left(\frac{1}{120}\frac{(3\delta-1)}{\omega^2-\omega^3-\delta\omega-\delta\omega^2}\right)\kappa_{n+\omega}+\left(-\frac{1}{120}\frac{(3\omega-1)}{\delta^2-\delta^3-\delta\omega-\omega\delta^2}\right)\kappa_{n+\delta}\right] \quad (8)$$
$$+h^4\left[\left(\frac{1}{120}\frac{(2\delta+2\omega-5\delta\omega-1)}{\delta+\omega-\delta\omega-1}\right)\kappa_{n+1}\right]$$

$$h^2t''_{n+1}=h^2\varphi''_n+h^3\varphi'''_n+h^4\left[\left(-\frac{1}{60}\frac{(5\delta+5\omega-20\delta\omega-2)}{\delta\omega}\right)\kappa_n\right]$$
$$-h^4\left[\left(\frac{1}{60}\frac{(5\delta-2)}{\omega^2-\omega^3-\delta\omega-\delta\omega^2}\right)\kappa_{n+\omega}\right] \quad (9)$$
$$h^4\left[+\left(-\frac{1}{60}\frac{((5\omega-2)}{\delta^2-\delta^3-\delta\omega-\omega\delta^2}\right)\kappa_{n+\delta}+\left(\frac{1}{60}\frac{(5\delta+5\omega-10\delta\omega-3)}{\delta+\omega-\delta\omega-1}\right)\kappa_{n+1}\right]$$

$$h^3t'''_{n+1}=h^3\varphi'''_n+h^4\left[\left(-\frac{1}{12}\frac{(2\delta+2\omega-6\delta\omega-1)}{\delta\omega}\right)\kappa_n\right]$$
$$-h^4\left[\left(\frac{1}{12}\frac{(2\delta-1)}{\omega^2-\omega^3-\delta\omega-\delta\omega^2}\right)\kappa_{n+\omega}+\left(-\frac{1}{12}\frac{((2\omega-1)}{\delta^2-\delta^3-\delta\omega-\omega\delta^2}\right)\kappa_{n+\delta}\right] \quad (10)$$
$$+h^4\left[\left(\frac{1}{12}\frac{(4\delta+4\omega-6\delta\omega-3)}{\delta+\omega-\delta\omega-1}\right)\kappa_{n+1}\right]$$

To obtain the values of the optimized points, we expand (10) using Taylor series around the point t'''_n to obtain the corresponding local truncation error as

$$\mathscr{L}(\varphi'''(t_{n+1}),h)=\frac{(5\delta+5\omega-10\delta\omega-3)\,\varphi^{(8)}(t_n)\,h^8}{1440}+\mathcal{O}(h^9) \quad (11)$$

We then optimize (11) by equating it to zero, taking δ as a free parameter by assigning the value $\delta=\frac{2}{3}$ in (11), we obtain. We then substitute the values of δ and ω into equation (4), and solve using Gaussian elimination method to obtain the block form of the equation in form of

$$A^{[i]}\varphi_m^{[i]}=A^{[i]}\varphi_m^{[i]}+\sum_{i=0}^{k}D^{[i]}G_m^{[i]},\quad i=\{0,\omega,\delta,N\} \quad (12)$$

when $i=0$,

$$\begin{bmatrix} \varphi_{n+\frac{1}{5}} \\ \varphi_{n+\frac{2}{3}} \\ \varphi_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \varphi_n \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ \frac{2}{3} \\ 1 \end{bmatrix} \begin{bmatrix} h\varphi'_n \end{bmatrix} + \begin{bmatrix} \frac{1}{50} \\ \frac{2}{9} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} h^2\varphi''_n \end{bmatrix} + \begin{bmatrix} \frac{1}{750} \\ \frac{4}{81} \\ \frac{1}{6} \end{bmatrix} \begin{bmatrix} h^3\varphi'''_n \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & \frac{1931}{39375000} \\ 0 & 0 & \frac{664}{229635} \\ 0 & 0 & \frac{23}{2520} \end{bmatrix} \begin{bmatrix} h^4 f_{n-\frac{1}{5}} \\ h^4 f_{n-\frac{2}{3}} \\ h^4 f_n \end{bmatrix} + \begin{bmatrix} \frac{221}{11025000} & -\frac{99}{30625000} & \frac{4}{4921875} \\ \frac{1700}{321489} & \frac{2}{59535} & \frac{4}{229635} \\ \frac{25}{882} & \frac{9}{1960} & -\frac{1}{2520} \end{bmatrix} \begin{bmatrix} h^4 f_{n+\frac{1}{5}} \\ h^4 f_{n+\frac{2}{3}} \\ h^4 f_{n+1} \end{bmatrix} \tag{13}$$

when $i = 1$,

$$\begin{bmatrix} h\varphi'_{n+\frac{1}{5}} \\ h\varphi'_{n+\frac{2}{3}} \\ h\varphi'_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h\varphi'_n \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ \frac{2}{3} \\ 1 \end{bmatrix} \begin{bmatrix} h^2\varphi''_n \end{bmatrix} + \begin{bmatrix} \frac{1}{50} \\ \frac{2}{9} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} h^3\varphi'''_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{113}{125000} \\ 0 & 0 & \frac{47}{3645} \\ 0 & 0 & \frac{1}{40} \end{bmatrix} \begin{bmatrix} h^4 f_{n-\frac{1}{5}} \\ h^4 f_{n-\frac{2}{3}} \\ h^4 f_n \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{29}{60000} & -\frac{9}{125000} & \frac{9}{500000} \\ \frac{25}{729} & \frac{1}{405} & -\frac{1}{3645} \\ \frac{25}{224} & \frac{9}{280} & -\frac{1}{480} \end{bmatrix} \begin{bmatrix} h^4 f_{n+\frac{1}{5}} \\ h^4 f_{n+\frac{2}{3}} \\ h^4 f_{n+1} \end{bmatrix} \tag{14}$$

when $i = 2$

$$\begin{bmatrix} h^2\varphi''_{n+\frac{1}{5}} \\ h^2\varphi''_{n+\frac{2}{3}} \\ h^2\varphi''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h^2\varphi''_n \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ \frac{2}{3} \\ 1 \end{bmatrix} \begin{bmatrix} h^3\varphi'''_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{881}{75000} \\ 0 & 0 & \frac{8}{243} \\ 0 & 0 & \frac{1}{24} \end{bmatrix} \begin{bmatrix} h^4 f_{n-\frac{1}{5}} \\ h^4 f_{n-\frac{2}{3}} \\ h^4 f_n \end{bmatrix} + \begin{bmatrix} \frac{8}{875} & -\frac{207}{175000} & \frac{11}{37500} \\ \frac{275}{1701} & \frac{2}{63} & -\frac{1}{243} \\ \frac{25}{84} & \frac{9}{56} & 0 \end{bmatrix} \begin{bmatrix} h^4 f_{n+\frac{1}{5}} \\ h^4 f_{n+\frac{2}{3}} \\ h^4 f_{n+1} \end{bmatrix} \tag{15}$$

when $i=3$

$$\begin{bmatrix} h^3\varphi'''_{n+\frac{1}{5}} \\ h^3\varphi'''_{n+\frac{2}{3}} \\ h^3\varphi'''_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h^3\varphi'''_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{253}{3000} \\ 0 & 0 & \frac{1}{81} \\ 0 & 0 & \frac{1}{24} \end{bmatrix} \begin{bmatrix} h^4 f_{n-\frac{1}{5}} \\ h^4 f_{n-\frac{2}{3}} \\ h^4 f_n \end{bmatrix} + \begin{bmatrix} \frac{209}{1680} & -\frac{81}{7000} & \frac{17}{6000} \\ \frac{250}{567} & \frac{5}{21} & -\frac{2}{81} \\ \frac{125}{336} & \frac{27}{56} & \frac{5}{48} \end{bmatrix} \begin{bmatrix} h^4 f_{n+\frac{1}{5}} \\ h^4 f_{n+\frac{2}{3}} \\ h^4 f_{n+1} \end{bmatrix} \qquad (16)$$

Equations (13) - (16) form the One-step Block Optimized Scheme (OBOS) derived for the approximation of fourth-order initial value problems (IVPs) of (1).

3. ANALYSIS OF BASIC PROPERTIES OF OBOS

3.1 Order and error constant

Let the linear difference operator $\mathscr{L}$ associated with the new method be defined by

$$\mathscr{L}[(\varphi(t),h)] = \sum_{j=0}^{N} \rho_j \varphi(t+jh) - h^4 \sum_{j=0}^{N} (\kappa_j \varphi^{(4)}(t+jh)) \qquad (17)$$

where $\varphi(t)$ is the exact solution satisfying equation (17) and can be written in Taylor's series expansion about the point t_n to obtain the expression

$$\mathscr{L}[(\varphi(t),h)] = \bar{c}_0\varphi(t) + \bar{c}_1 h\varphi'(t) + \bar{c}_2 h^2\varphi''(t) + \cdots + \bar{c}_{p+4} h^{p+4}\varphi^{(p+4)}(t) + \cdots \qquad (18)$$

Similarly, the new optimized fourth order hybrid methods (OBOS) is of order p if,

$$\mathscr{L}[(\varphi(t),h)] = \mathscr{O}(h^{p+4}), \quad \bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \cdots = \bar{c}_{p+3} = 0, \ \bar{c}_{p+4} \neq 0$$

Therefore, the principal local truncation error $t_n + N$ is then defined to be

$$\bar{c}_{p+4} h^{p+4} \varphi^{(p+4)}(t_n)$$

where

$$\begin{aligned} c_0 &= \sum_{j=0}^{N} \rho_j \\ c_1 &= \sum_{j=1}^{N} j\rho_j - \sum_{j=0}^{N} \kappa_j \\ c_2 &= \frac{1}{2}\sum_{j=1}^{N} j^2\rho_j - \sum_{j=0}^{N} j\kappa_j \qquad (19) \\ &\vdots \\ c_q &= \frac{1}{q!}\sum_{j=1}^{N} j^q\rho_j - \frac{1}{(q-1)!}\sum_{j=0}^{N} j^{q-1}\kappa_j - \frac{1}{(q-2)!}\sum_{j=0}^{N} j^{q-2}\psi_j \quad q = 2,3,4,\ldots \end{aligned}$$

Hence, our methods has order
$c_{p+4}=8$, in which $c_0=c_1=c_2=c_3=c_4=\cdots=c_7$ with c_8
as the truncation error. Therefore, $p=(4,4,4,4,4,4,4,4,4,4,4,5)^T$. The error constants and order (p) of the new method (OBOS) are shown in Table below:

Table 1. Error constants and order, p, of OBOS

S/N	**Scheme**	**Order**	**Error constants** (C_{p+1})
1	$\varphi_{n+\frac{1}{5}}$	4	$-\frac{403}{47250000000}$
2	$\varphi_{n+\frac{2}{3}}$	4	$-\frac{4}{10333575}$
3	φ_{n+1}	4	$\frac{1}{604800}$
4	$\varphi'_{n+\frac{1}{5}}$	4	$\frac{1}{604800}$
5	$\varphi'_{n+\frac{2}{3}}$	4	$\frac{1}{2296350}$
6	φ'_{n+1}	4	$\frac{1}{75600}$
7	$\varphi''_{n+\frac{1}{5}}$	4	$-\frac{203}{67500000}$
8	$\varphi''_{n+\frac{2}{3}}$	4	$\frac{7}{328050}$
9	φ''_{n+1}	4	$\frac{1}{21600}$
10	$\varphi'''_{n+\frac{1}{5}}$	4	$-\frac{4}{140625}$
11	$\varphi'''_{n+\frac{2}{3}}$	4	$\frac{1}{7290}$
12	φ'''_{n+1}	5	$-\frac{1}{108000}$

3.2 Zero-stability

A linear multistep method is said to be zero-stable if the roots Z_i, $i=1,2,\ldots,k$ (grid and non-grid points) of the first characteristics polynomial defined by $\rho(z)=det(zA^0-D)$ satisfies $|z_i|<1$ and the root $|z|=1$ having multiplicity not exceeding one (Lambert [12]). For our method,

$$\rho(z) = \left[z \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&1&0&0&\frac{1}{5}h&0&0&\frac{1}{50}h^2&0&0&\frac{1}{750}h^3 \\ 0&0&1&0&0&\frac{2}{3}h&0&0&\frac{2}{9}h^2&0&0&\frac{4}{81}h^3 \\ 0&0&1&0&0&h&0&0&\frac{1}{2}h^2&0&0&\frac{1}{6}h^3 \\ 0&0&0&0&0&1&0&0&\frac{1}{5}h&0&0&\frac{1}{50}h^2 \\ 0&0&0&0&0&1&0&0&\frac{2}{3}h&0&0&\frac{2}{9}h^2 \\ 0&0&0&0&0&1&0&0&h&0&0&\frac{1}{2}h^2 \\ 0&0&0&0&0&0&0&0&1&0&0&\frac{1}{4}h \\ 0&0&0&0&0&0&0&0&1&0&0&\frac{2}{3}h \\ 0&0&0&0&0&0&0&0&1&0&0&h \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \right] = 0$$

$$= z^8 (z-1)^4 = 0$$

Finding its determinant and solving the characteristic equation gives

$$\rho(z) = z^8 (z-1)^4 = 0, \quad z = 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1$$

Since, $|z_i| = |0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1| \leq 1$, the method is zero-stable.

3.3 Consistency

The method, (OSOB) is consistent since it has order $p = 4 \geq 1$.

3.3 Convergence

According to Awoyemi [7], Eq. (5) converges if it is zero stable and consistent. Hence, the derived method converges.

4. NUMERICAL RESULTS

The following problems are solved in order to examine the validity of the new block method (OBOS). All computations are done using MAPLE 18.

The following notations are used in the result Tables:

RAYMOND1 = Error in Raymond et al in prob 1

AKINNUKAWE = Error in Akinnukawe et al in prob 2

RAYMOND2 = Error in Raymond et al in prob 3

EOBOS = Error in One-step Block Optimized Scheme

OBOS = One-step Block Optimized Scheme

where

$$\text{Error}^{(i)} = \left| \varphi^{(i)}_{\text{exact}} - \varphi^{(i)}_{\text{approx.}} \right|$$

Four problems are considered.

Problem 1 [8]

$$\varphi^{iv} - 4\varphi''' + 6\varphi'' + \varphi' + \varphi = 0, \quad \varphi(0) = 0, \varphi'(0) = 1, \varphi''(0) = 0, \varphi'''(0) = 1$$

with exact solution given by

$$\varphi(t) = t\exp(t) - t^2\exp(t) + \frac{2}{3}t^3\exp(t)$$

Problem 2 [9]

$$\varphi^{(iv)}(t) + \varphi''(t) = 0, \quad 0 \le t \le \frac{\pi}{2};$$

$$\varphi(0) = 0,$$

$$\varphi'(0) = -\frac{1\cdot 1}{72 - 50\pi},$$

$$\varphi''(0) = \frac{1}{144 - 100\pi},$$

$$\varphi'''(0) = \frac{1\cdot 2}{144 - 100\pi}$$

The exact solution is $\varphi(t) = \dfrac{1 - t - \cos(t) - 1\cdot 2\sin(t)}{144 - 100\pi}$

Problem3 [8]

Consider the fourth order initial value problem

$$\varphi^{iv} = -\left[3\varphi'' + (2 + \rho\cos(\gamma t))\right], \quad \varphi(0) = 1, \varphi'(0) = \varphi''(0) = \varphi'''(0) = 0$$

The exact solution is given by $\varphi(t) = 2\cos(t) - \cos(t\sqrt{2})$

Table 2. Comparison of results and errors for Problem 1 [8]

t	φ_{exact}	OBOS	EOBOS	RAYMOND1[8]
0.1	0.010000170020911319881	0.01000017002091131985	3.10e-20	2.55208e-12
0.2	0.020001387338337889911	0.02000138733833788919	7.21e-19	3.64210e-12
0.3	0.030004775119658503801	0.03000477511965850086	2.941e-18	4.5313e-12
0.4	0.040011541655353249807	0.04001154165535324217	7.637e-18	1.3406e-12
0.5	0.050022983002559143889	0.05002298300255912811	1.5779e-17	3.28547e-12
0.6	0.060040485687860814478	0.06004048568786078607	2.8407e-17	4.59125e-12
0.7	0.070065529470429623624	0.07006552947042957699	4.6634e-17	5.47318e-12
0.8	0.080099690166643335457	0.08009969016664326386	7.1597e-17	1.96524e-12
0.9	0.090144642537337460546	0.09014464253733735601	1.0453e-16	2.34526e-12
1.0	0.100202163238858770088	0.10020216323885857126	1.4672e-16	2.55587e-12

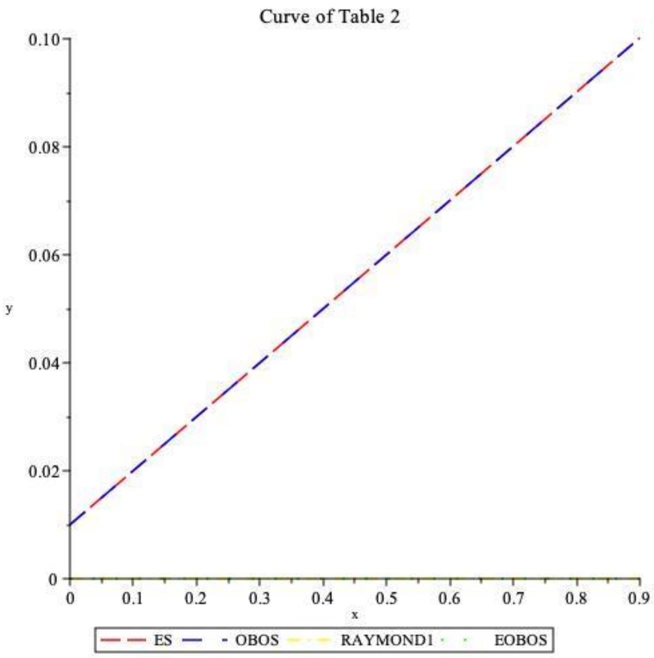

Fig. 1. Curves of Table 2

Table 3. Comparison of results and errors for Problem 2 [9]

t	φ_{exact}	**OBOS**	**EOBOS**	**AKINNUKAWE[9]**
0.01	0.00012889983466749742	0.00012889983466749742	0.0000	3.80555e-17
0.02	0.00025720540846480060	0.00025720540846480060	0.0000	2.69424e-17
0.03	0.00038490976337573504	0.00038490976337573504	0.0000	6.55942e-17
0.04	0.00051200600150551386	0.00051200600150551386	0.0000	3.20924e-17
0.05	0.00063848728577052155	0.00063848728577052155	0.0000	3.14419e-17
0.06	0.00076434684058201660	0.00076434684058201660	0.0000	3.22008e-17
0.07	0.00088957795252368476	0.00088957795252368476	0.0000	3.45860e-17
0.08	0.00101417397102297505	0.00101417397102297505	1.0000e-20	2.51535e-17
0.09	0.00113812830901615149	0.00113812830901615149	1.0000e-20	6.87384e-17
0.10	0.00126143444360699398	0.00126143444360699398	2.0000e-20	1.56125e-16

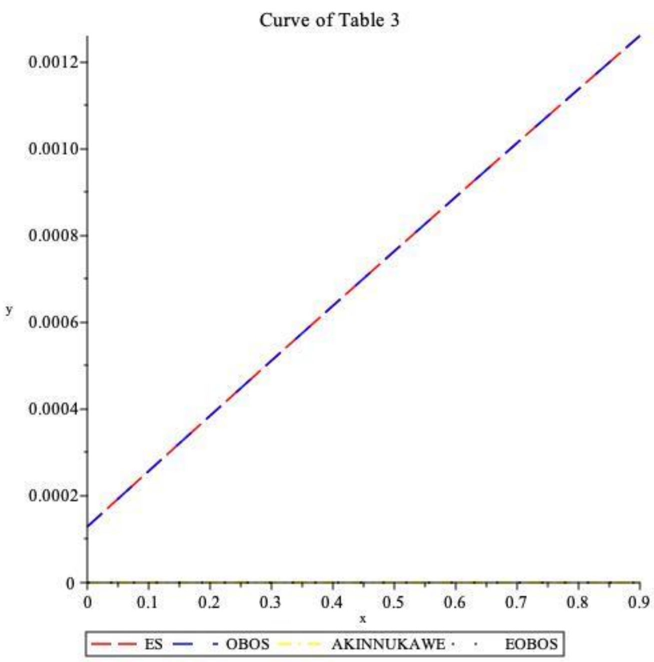

Fig. 2. Curves of Table 3

Table 4. Comparison of results and errors for Problem 3 [8]

t	φ_{exact}	OBOS	EOBOS	RAYMOND2[8]
0.003125	0.9999999999920527211	0.99999999999205272179	2.00e-20	1.0434e-14
0.006250	0.9999999998728439213	0.99999999987284392119	4.00e-20	9.8731e-14
0.009375	0.9999999993562754943	0.99999999935627549420	7.00e-20	3.1317e-
0.015625	0.99999999796552658061	0.99999999796552658061	0.0000	6.6668e-13
0.015625	0.9999999950330675336	0.99999999503306753346	0.0000	1.1507e-12
0.018750	0.99999998970067947566	0.99999998970067947566	8.00e-20	1.7445e-12
0.021875	0.99999998091947944409	0.99999998091947944409	5.00e-20	2.4220e-12
0.025000	0.99999996744995111885	0.99999996744995111891	6.00e-20	3.1554e-12
0.028125	0.99999994786198113954	0.99999994786198113957	3.00e-20	3.9178e-12
0.031250	0.99999992053490100510	0.99999992053490100510	6.00e-20	4.6852e-12

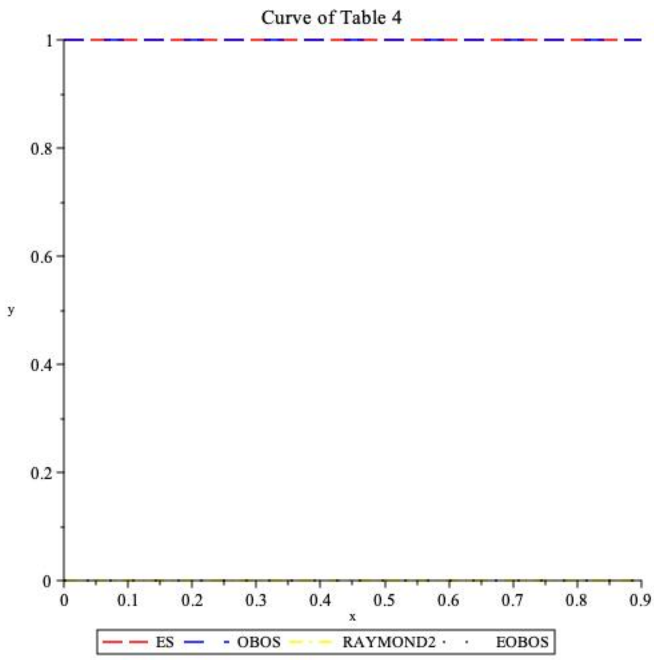

Fig. 3. Curves of Table 3

5. DISCUSSION

In Table 2, exact solution, computed solutions and errors of OBOS and Raymond *et al*, [8] for solving Problem 1 are shown. The results from table 2 as well as Fig.(1) reveal the efficiency and superiority of the new scheme (OBOS) over EIR1. In addition, in Tables 3 and 4, as well as in Fig. 2 and Fig. 3, exact and computed solutions of the newly developed method, OBOS, for the solution of Problems 2 and 3, are demonstrated. The new method outperforms method proposed by Akinnukawe *et al*, [9] in terms of accuracy. Similarly, the numerical results in Table 4 show that the new block method (OBOS) is an effective and efficient scheme for solving initial value problems of fourth-order.

6. CONCLUSION

In this paper, an efficient optimized hybrid one-step scheme for solving fourth-order initial value problems (IVPs) via interpollation and collocation methods was developed and implemented. Approximated power series is used as the basis function with its fourth derivatives as collocating equation. The analyses of the method were shown to be consistent, convergent and zero stable. The new scheme has proved to be an effective and efficient solution to fourth-order ODEs as shown in Tables 1, 2 and 3 and in Figures 1 - 3.

REFERENCES

1. Areo, E. A., and Omole, E. O. (2015). Half-step symmetric continuous hybrid block method for numerical solution of fourth-order ODEs. Archives of Applied Science Research, 7(39).
2. Butcher, J.C. (2008). Numerical methods for ordinary differential equations (2nd ed.), John Wiley and Sons, Ltd.
3. Ndanusa, A., Tafida, F.U. (2016). Predictor – corrector methods of high order for numerical integration of initial value problems. International Journal of Scientific and Innovative Mathematical Research, 4(2), 47-55.
4. Kuboye, J. O., Omar, Z. (2015). New Zero-stable block method for direct solution of fourth-order

ODEs. Indian Journal of Science and Technology, 8(1).

5. Awoyemi, D.O. (2005). Algorithmic collocation approach for direct solution of fourth-order initial value problems of ordinary differential equations. International Journal of Computer Mathematics, 80(8), 321-329.
6. Fatunla, S.O. (1991). Block method for second-order IVPs. *International journal of computer mathematics*, *41*(1-2), 55-63.
7. Awoyemi, D. O. (1999). A class of continuous methods for general second order initial value problems in ordinary differential equations. *International journal of computer mathematics*, *72*(1), 29-37.
8. Raymond, D., Kyagya, Y.T., Sabo, J., & Lydia, A. (2023). Numerical application of higher order linear block method for solving some fourth order initial value problems. *African Scientific Reports, 67,* 1-10. doi={https://doi.org/10.4648/asr.2022.2.1.67}
9. Akinnukawe, B.I., Kuboye, J.O., Okunuga, S.A., Lydia, A. (2023). Block bi-basis collocation method for direct approximation of fourth-order initial value problems. Journal of the Nigerian Mathematical Society, 41(1), 1-18.
10. Akinnukawe, B. I., & Odekunle, M. R. (2023). Block Bi-Basis Collocation Method for Direct Approximation of Fourth-order IVP: Block Bi-Basis Collocation Method for. *Journal of the Nigerian Mathematical Society*, *42*(1), 1-18.
11. Kuboye, J., Elusakin, O. R., & Quadri, O. F. (2020). Numerical algorithms for direct solution of fourth order ordinary differential equations. *Journal of the Nigerian Society of Physical Sciences*, 218-227.
12. Lambert, J.D. (1973). Computational mathematics in ordinary differential equations. John Wiley and Sons, Inc.