Engineering and Technology Journal e-ISSN: 2456-3358

Volume 10 Issue 02 February-2025, Page No.- 3851-3856

DOI: 10.47191/etj/v10i02.14, I.F. – 8.482

Analysis of Nonlinear Vibration Arising in Micro-Electromechanical System

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ABSTRACT: In this work, the nonlinear vibration arising in the micro-electromechanical system is investigated by using the equivalent linearization method based on a weighted averaging concept. The analytical solution of the system is carried out, and the relationship between the frequency and the initial amplitude is established in a closed analytical form. In order to verify the accuracy of the present method, some illustrative examples are analyzed in detail and the results are compared with other analytical and numerical solutions.

KEYWORD: equivalent linearization method, nonlinear oscillator, micro-electromechanical systems.

1. INTRODUCTION

Micro-beams are major parts of micro-electromechanical systems (MEMS). In a MEMS device, besides the fixed electrodes, the movable components are often modeled as micro-beams. The applied voltage causes an electrostatic force to act on the micro-beams, making them to vibrate. MEMS devices including micro-switches, micro-actuators, micro-sensors,... have many applications in the high-tech fields such as electronics engineering, aerospace engineering, biomedical engineering and optical engineering [1].

While working, electrostatic actuation, large deflection and damping caused by different sources give rise to nonlinear behavior. Nonlinearity in MEMS may cause some difficulties in computations. Until now, several techniques have been used to find numerical solutions, for example, the shooting method [2], the differential quadrature method [3] and the Adomian decomposition method [4]. Although it is difficult to obtain analytical approximations for different phenomena in MEMS, there are some analytical techniques for nonlinear problems of MEMS such as perturbation techniques [5], the energy balance method [6], the homotopy analysis method [7] and the He's Variational Approach [8].

The equivalent linearization method is one of the common approaches to an approximate analysis of dynamical systems. The original linearization for deterministic systems was proposed by Krylov and Bogoliubov [9]. Then Caughey [10] expanded this method for stochastic systems. However, the accuracy of the equivalent linearization method with conventional averaging normally reduces for middle or strong nonlinear systems. Thus, this method has been developed by many authors [11, 12]. Recently, Anh [13] has developed this
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method in the following way: instead of applying the conventional averaging method, the author has introduced weighted coefficient functions and averaging values are calculated in a new way which is called the weighted averaging values. This proposed method was then applied by Anh et al. [14] to analyze some strong nonlinear oscillations. In this work, this method will be applied to analyze a nonlinear vibration arising in the micro-electromechanical system.

2. MODELING AND FORMULATION

Consider a fully clamped micro-beam placed between two stationary electrodes with length *L*, thickness *h* and width *b* whose sketch is shown in Figure 1. Employing the classical beam theory and taking into account of the mid-plane stretching effect as well as the distributed electrostatic force, the following dimensionless equation of motion for the micro-beam can be formulated via the Galerkin method [6, 7]:

$$(a_1u^4 + a_2u^2 + a_3)\ddot{u} + a_4u + a_5u^3 + a_6u^5 + a_7u^7 = 0 \quad (1)$$

where *u* is the dimensionless deflection of the micro-beam, a dot denotes the derivative with respect to the dimensionless time variable $\tau = t\sqrt{\bar{E}I/(\rho bhl^4)}$, in which *I* and *t* are the second moment of area of the micro-beam cross-section and time, respectively; ρ is the mass density of the micro-beam; $\bar{E} = E/(1-\nu^2)$ is the effective modulus depending on Young's modulus (*E*) and Poisson's ratio (ν) of the micro-beam. The complete formulation of Eq. (1) can be referenced from Refs. [6, 7] for details and the expressions of the parameters a_i (*i=1-7*) are presented as below

$$
\begin{aligned}
&a_1=\int_0^1 \phi^6 d\xi;\ a_2=-2\int_0^1 \phi^4 d\xi;\ a_3=\int_0^1 \phi^2 d\xi;\\
&a_4=\int_0^1\left(\phi''''\phi-N\phi''\phi-V^2\phi^2\right)d\xi;\\
&a_5=-\int_0^1\left(2\phi''''\phi^3-2N\phi''\phi^3+\alpha\phi''\phi\int_0^1\left(\phi'\right)^2 d\xi\right)d\xi;\\
&a_6=\int_0^1\left(\phi''''\phi^5-N\phi''\phi^5+2\alpha\phi''\phi^3\int_0^1\left(\phi'\right)^2 d\xi\right)d\xi;\\
&a_7=-\int_0^1\left(\alpha\phi''\phi^5\int_0^1\left(\phi'\right)^2 d\xi\right)d\xi.
\end{aligned}
\qquad (2)
$$

in which

$$
\alpha=\frac{6g_0^2}{h^2},\quad N=\frac{\bar{N}L^2}{\bar{E}I},\quad V^2=\frac{24\varepsilon_0 L^4\bar{V}^2}{\bar{E}h^3g_0^3},\quad \xi=\frac{x}{L}, \qquad (3)
$$

where a prime (') indicates the partial derivative with respect to the coordinate variable ξ. In Eq. (3), the parameters $\bar{N}, \bar{V}, \varepsilon_0$ and g_0 are, respectively, the applied axial force, applied electrostatic voltage, vacuum permittivity and initial gap between the micro-beam and the fixed substrates. Owing to the micro-beam is fully fixed at both the ends, the trial function $\phi(\xi)$ in Eq. (2) can be is chosen as follows [6, 7]:

$\phi(\xi)=16\xi^2(1-\xi)^2$.

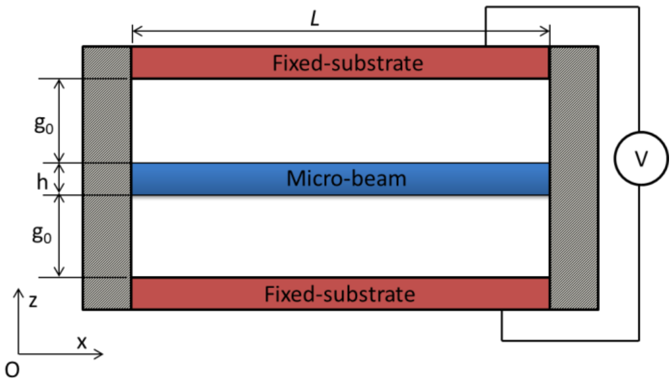


Figure 1: Geometry for an electrostatically actuated micro-beam with fixed substrates

In the next section, the equivalent linearization method with a weighted averaging is employed to study this nonlinear vibration.

3. APPLICATION OF THE EQUIVALENT LINEARIZATION METHOD WITH A WEIGHTED AVERAGING

At first, Eq. (1) is solved by applying the equivalent linearization method. The linearized equation of Eq. (1) is taken as:

$$\ddot{u}+\omega^2 u=0. \qquad (4)$$

The equation error between Eq. (1) and Eq. (4) is:

$$e(u)=(a_1u^4+a_2u^2+a_3)\ddot{u}+a_4u+a_5u^3+a_6u^5+a_7u^7-\ddot{u}-\omega^2 u. \qquad (5)$$

There are some criteria for determining the coefficient ω^2 in Eq. (4). However, the most common criterion is the mean square error criterion:

$$\left\langle e^2(u)\right\rangle \rightarrow \underset{\omega^2}{Min}. \qquad (6)$$

Thus, from the condition:

$$\frac{\partial}{\partial\omega^2}\left\langle e^2(u)\right\rangle=0,$$

it leads to:

$$\omega^2 = \frac{a_1\langle u^5\ddot{u}\rangle + a_2\langle u^3\ddot{u}\rangle + a_3\langle u\ddot{u}\rangle - \langle u\ddot{u}\rangle}{\langle u^2\rangle} + \frac{a_4\langle u^2\rangle + a_5\langle u^4\rangle + a_6\langle u^6\rangle + a_7\langle u^8\rangle}{\langle u^2\rangle}. \tag{7}$$

The periodic solution of Eq. (4) has a form:

$$u(\tau) = A\cos(\omega\tau). \tag{8}$$

Here, let's consider the weighted coefficient proposed by Anh as follows [13]:

$$h(\tau) = s^2\omega^2\tau e^{-s\omega\tau},\ s>0, \tag{9}$$

The weighted coefficient (9), obtained as a product of the optimistic weighted coefficient τ and the pessimistic weighted coefficient $e^{-s\omega\tau}$, has one maximal value at $\tau_{max}=1/(\omega s)$, and then decreases to zero as $\tau \to \infty$. The properties of the weighted coefficient $h(\tau)$ in Eq. (9) can be reviewed in Refs. [13, 14]. Based on the weighted coefficient (9), the new weighted averaging value is proposed [13]:

$$\langle u(\omega\tau)\rangle = \int_0^{+\infty} s^2\omega^2\tau e^{-s\omega\tau}u(\omega\tau)d\tau = \int_0^{+\infty} s^2\tau' e^{-s\tau'}u(\tau')d\tau',\ \tau' = \omega\tau. \tag{10}$$

With the periodic solution given in Eq. (8), and by using Eq. (10), we can calculate the averaging values in Eq. (7) as follows:

$$\langle u^2\rangle = \langle A^2\cos^2(\omega\tau)\rangle = \int_0^{+\infty} A^2\cos^2(\omega\tau)s^2\omega^2\tau e^{-s\omega\tau}d\tau = \int_0^{+\infty} A^2\cos^2(\tau')s^2\tau' e^{-s\tau'}d\tau' = A^2\frac{s^4+2s^2+8}{(s^2+4)^2}, \tag{11}$$

similarly, we get:

$$\langle u^4\rangle = \langle A^4\cos^4(\omega\tau)\rangle = A^4\frac{248s^4+416s^2+1536+28s^6+s^8}{(s^2+4)^2(s^2+16)^2}, \tag{12}$$

$$\langle u^6\rangle = \langle A^6\cos^6(\omega\tau)\rangle = A^6\frac{\begin{pmatrix}1658880+440064s^2+282496s^4+\\+45712s^6+3168s^8+94s^{10}+s^{12}\end{pmatrix}}{(s^2+4)^2(s^2+16)^2(s^2+36)^2}, \tag{13}$$

$$\langle u^8\rangle = \langle A^8\cos^8(\omega\tau)\rangle = A^8\frac{\begin{pmatrix}1516142592s^2+1014806528s^4+192596992s^6+17013120s^8+\\+5945425920+768000s^{10}+18256s^{12}+216s^{14}+s^{16}\end{pmatrix}}{(s^2+4)^2(s^2+16)^2(s^2+36)^2(s^2+64)^2}, \tag{14}$$

$$\langle u\ddot{u}\rangle = -\langle A^2\omega^2\cos^2(\omega\tau)\rangle = -A^2\omega^2\frac{s^4+2s^2+8}{(s^2+4)^2}, \tag{15}$$

$$\langle u^3\ddot{u}\rangle = -\langle A^4\omega^2\cos^4(\omega\tau)\rangle = -A^4\omega^2\frac{248s^4+416s^2+1536+28s^6+s^8}{(s^2+4)^2(s^2+16)^2}, \tag{16}$$

$$\langle u^5\ddot{u}\rangle = -\langle A^6\omega^2\cos^6(\omega\tau)\rangle = -A^6\omega^2\frac{\begin{pmatrix}1658880+440064s^2+282496s^4+\\+45712s^6+3168s^8+94s^{10}+s^{12}\end{pmatrix}}{(s^2+4)^2(s^2+16)^2(s^2+36)^2}, \tag{17}$$

Substituting Eqs. (11)-(17) into Eq. (7), the approximate frequency can be obtained as:

$$\omega = \sqrt{\frac{a_4+a_5A_2(s)A^2+a_6A_1(s)A^4+a_7A_3(s)A^6}{a_1A_1(s)A^4+a_2A_2(s)A^2+a_3}}, \tag{18}$$

where:

$$A_1(s) = \frac{1658880+440064s^2+282496s^4+45712s^6+3168s^8+94s^{10}+s^{12}}{(s^4+2s^2+8)(s^2+16)^2(s^2+36)^2}, \tag{19}$$

$$A_2(s) = \frac{248s^4+416s^2+1536+28s^6+s^8}{(s^4+2s^2+8)(s^2+16)^2}, \tag{20}$$

$$A_3(s) = \frac{\begin{pmatrix}1516142592s^2+1014806528s^4+192596992s^6+17013120s^8+\\+5945425920+768000s^{10}+18256s^{12}+216s^{14}+s^{16}\end{pmatrix}}{(s^4+2s^2+8)(s^2+16)^2(s^2+36)^2(s^2+64)^2}, \tag{21}$$

According to Eqs. (8) and (18), we can obtain the following approximate solution:

$$u(\tau) = A\cos\left(\sqrt{\frac{a_4 + a_5 A_2(s)A^2 + a_6 A_1(s)A^4 + a_7 A_3(s)A^6}{a_1 A_1(s)A^4 + a_2 A_2(s)A^2 + a_3}}\ \tau\right). \tag{22}$$

It is noted that the approximation frequency (18) depends not only on the parameters of the problems a_1, a_2, a_3, a_4, a_5, a_6 and a_7 but also on the initial amplitude A and tuning parameter s.

4. NUMERICAL RESULTS AND DISCUSSION

In order to show the efficiency of the proposed method, the results obtained in this work are compared with those obtained by using the energy balance method (EBM) [6] with different values of A, N, α and V. Comparing the approximate frequencies obtained by two analytical methods are shown in Table 1. It can be observed the accuracy of the current results compared with the results obtained by the energy balance method [6]. In this comparison, several values of the tuning parameter s are selected as s = 1, 2, 3 and 4.

Table 1: Comparison of frequencies corresponding to various parameters of system

case	A	N	A	V	ω_{exact} [7]	ω_{EBM} [6]	$\omega_{present}$ $s=1$	$s=2$	$s=3$	$s=4$
1	0.3	10	24	0	26.8372	26.3867	26.7089	26.7577	26.9112	27.0291
						(1.68%)*	(0.48%)	(0.29%)	(0.28%)	(0.71%)
2	0.3	10	24	20	16.6486	16.3829	16.5431	16.5865	16.7173	16.8165
						(1.51%)	(0.63%)	(0.37%)	(0.41%)	(1.01%)
3	0.6	10	24	10	28.5382	26.5324	28.0471	28.2199	28.6501	28.9704
						(7.03%)	(1.72%)	(1.12%)	(0.39%)	(1.46%)
4	0.6	10	24	20	18.5902	17.5017	18.4460	18.5507	18.6938	18.7690
						(5.86%)	(0.78%)	(0.21%)	(0.54%)	(0.83%)

* difference from the exact frequency

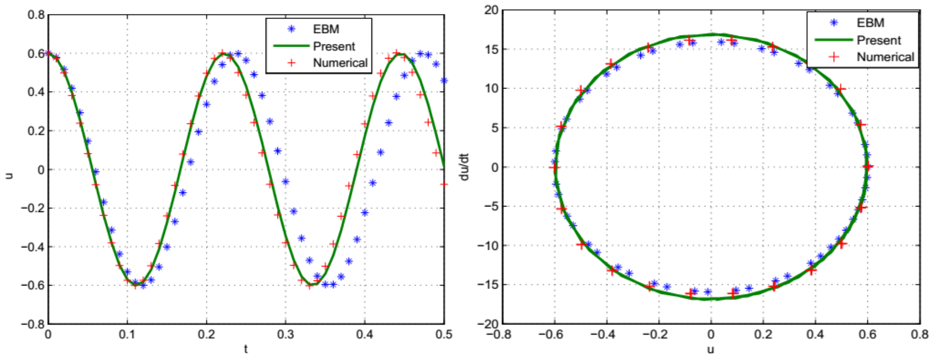
It can be observed from Table 1 that for $s = 2$ and $s = 3$ the current method gives very accurate results. From Eqs. (18) and (22), in case of $s = 2$, the approximate expressions of frequency and solution, respectively, can be obtained as below:

$$\omega = \sqrt{\frac{a_4 + 0.72a_5A^2 + 0.575a_6A^4 + 0.4836a_7A^6}{0.575a_1A^4 + 0.72a_2A^2 + a_3}}, \tag{23}$$

and

$$u(\tau) = A\cos\left(\sqrt{\frac{a_4 + 0.72a_5A^2 + 0.575a_6A^4 + 0.4836a_7A^6}{0.575a_1A^4 + 0.72a_2A^2 + a_3}}\ \tau\right). \tag{24}$$

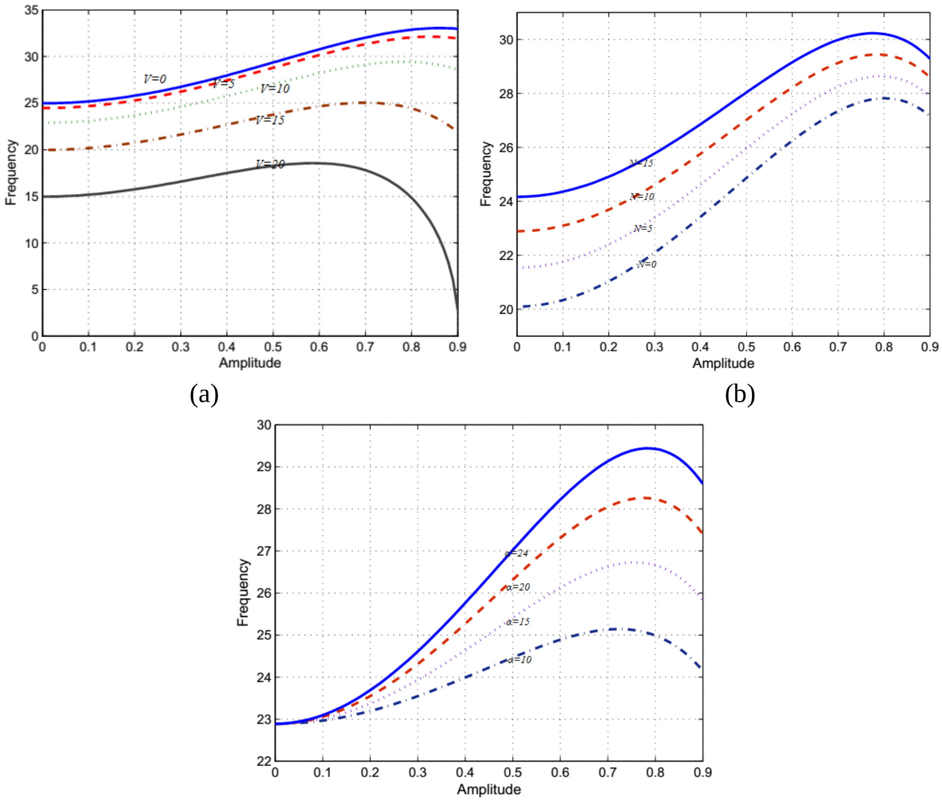
The comparison of the analytical solution $u(\tau)$ obtained in Eq. (24) with the analytical solution achieved by the energy balance method [6] and the numerical solution is presented in Figure 2. It can be observed from this figure that the motion of the system is a periodic motion and the vibration amplitude is a function of the initial condition. Also, the accuracy of the obtained analytical solution can be observed.



(a) (b)

Figure 2: Comparison of solutions corresponding to various parameters of system: *(a) A=0.3, N=10, α=24, V=0; (b) A=0.6, N=10, α=24, V=10.*

The effects of V, N and α parameters on the nonlinear frequency are presented in Figure 3. It can be observed from Figure 3 that the nonlinear frequency of the micro-beam decreases as the applied voltage (V) increases, and increases as the axial compressive force (N) and initial gap between the micro-beam and the fixed substrates (α) increase. However, when value of the applied voltage V is large, the micro-beam will be unstable, i.e., when V exceeds the critical value V_{cr}, the frequency of the micro-beam will approach to zero.



(c)

Figure 3: The effects of *V, N, α* parameter on the nonlinear frequency: (a) *α=24, N=10; (b) α=24, V=10; (c) N=10, V=10*

5. CONCLUSIONS

In this paper, the equivalent linearization method with a weighted averaging is applied to analyze the nonlinear vibration arising in the micro-electromechanical system. The technique is considered as an approach to develop the classical equivalent linearization method. This method inherits the convenience of the equivalent linearization method combining with the advantage of the weighted averaging operation. The relationship between frequency and amplitude is established. The accuracy of the solution is compared with the previously published results. Moreover, the effects of parameters of the system are also investigated. The results indicate that the solution procedure is easy to apply and can provide a remarkable accuracy.

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