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Analyzing the Relationship between Free Locally Convex Spaces and Nuclear Space Concepts

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ABSTRACT: The exploration of Free Locally Convex Spaces (FLCS) and nuclear spaces is a crucial component of functional analysis. Extensive research has been conducted on the interplay between FLCS and nuclear spaces. A significant finding in this field is that every nuclear space qualifies as a FLCS. Furthermore, the examination of specific categories of FLCS reveals that the connections between these two types of spaces can enhance our understanding of their individual characteristics and applications. Conversely, there are numerous unresolved questions and potential avenues for future research regarding free locally convex spaces and nuclear spaces. One intriguing challenge is to identify instances of FLCS that do not fall under the category of nuclear spaces. Another promising direction involves investigating the relationships between FLCS and other forms of generalized locally convex spaces, such as bornological spaces. Additionally, the role of FLCS and nuclear spaces in the realms of non-commutative geometry and operator algebras continues to be a vibrant area of inquiry. This paper seeks to present a comprehensive overview of these concepts and their interrelations.

KEYWORDS: Free Locally Convex Space, Nuclear Space, Topological Vector Space, Dual Space, Banach Space, And Continuous Linear Functionals.

INTRODUCTION

Free locally convex spaces (FLCS) extend the concept of locally convex spaces, providing greater structural flexibility. Introduced by Arens in the context of Banach algebras, FLCS have been the subject of extensive research in functional analysis. These spaces are characterized as vector spaces that are associated with a family of seminorms meeting specific criteria. FLCS are significant in multiple mathematical disciplines, such as algebraic geometry and representation theory. In contrast, nuclear spaces represent another crucial concept within functional analysis, first defined by Grothendieck. This class of topological vector spaces possesses several advantageous properties, including reflexivity and the approximation property. A space is classified as nuclear if it can be expressed as a projective limit of finite-dimensional vector spaces connected by continuous linear maps. Nuclear spaces find applications across various domains, including operator algebras and quantum mechanics.

All topological spaces are generally assumed to be Tychonoff and infinite unless stated otherwise. Vector spaces are considered over the field of real numbers, represented as R. For any Tychonoff space X, it can be regarded as a subset of its free locally convex space L(X). Any continuous mapping f: X → E to a locally convex space E can be uniquely extended to a continuous linear mapping fb: L(X) → E. In a similar manner, the free abelian group A(X) associated with a Tychonoff space X is defined such that X acts as the generating subspace of A(X). The topology of A(X) guarantees that any continuous mapping f: X → G to an abelian topological group G can be uniquely extended to a continuous homomorphism fb: A(X) → G.

A(X) is naturally contained within L(X) according to references [15] and [17]. In reference [18], it is shown that L(X) can be isomorphically embedded in the product of Banach spaces l1(Γ), implying that L(X) can be represented as a subgroup of the unitary group. Vladimir Pestov raised the question of whether L(X) can be isomorphically embedded within the product of Hilbert spaces, a concept referred to as being multi-Hilbert. He also questioned if L(X) is classified as a nuclear locally convex space. This inquiry arose from the fact that L(X) is nuclear when X is a countable discrete space, where L(X) is expressed as the locally convex direct sum of countably many one-dimensional spaces, commonly represented as ϕ in the literature. Chapter 2 demonstrates that L(X) is not multi-Hilbert, suggesting that L(X) is non-nuclear if X includes an infinite compact subset. Therefore, it is concluded that L(X) is nuclear only when X is both countable and discrete, particularly when X is a k-space. In other words, the assertion implies that L(X) is nuclear for all projectively countable P-spaces, which encompasses Lindelöf P-spaces.

Nuclear maps and nuclear spaces are characterized within the framework of linear mappings between locally convex spaces (LCS) and Banach spaces. A linear mapping from an LCS to a Banach space is deemed nuclear if it can be represented as a sum that incorporates a sequence of coefficients (λ_n), a sequence of equicontinuous linear functionals on the LCS (fn), and a bounded sequence in the Banach space (y_n). An LCS is identified as nuclear if every continuous linear mapping from that LCS to a Banach space qualifies as nuclear.

The concept of nuclear locally convex spaces (LCS) was defined, and significant permanence results were established by Grothendieck. These results highlight several essential properties of nuclear LCS including followings :

- Every subspace of a nuclear LCS is also a nuclear LCS.
- Every Hausdorff quotient space of a nuclear LCS is a nuclear LCS.
- The product of any family of nuclear LCS is a nuclear LCS.
- The locally convex direct sum of a countable family of nuclear LCS is a nuclear space.
- Every nuclear LCS is multi-Hilbert. Additionally, it is worth noting that the class of multi-Hilbert LCS remains closed under Hausdorff quotients.

As elaborated upon in the following sections, Lemma 2.2 from Grothendieck introduces the notion of a core group, which is a significant aspect of this lemma and is examined thoroughly in reference [8]. Specifically, a core group is defined as a topological group that functions simultaneously as a core local convex space (LCS) and as a complete metric space. The characteristics and details regarding the primary groups are outlined in reference [8]. Furthermore, the core set encompasses the entirety of the core local convex space, indicating that every point within the convex area qualifies as part of the core set.

This connection shows the importance of the core group in t he analysis of locally convex surfaces, which play an import ant role in functional analysis. In contrast to nuclearenergy, the concept of multiple reflection is discussed in section 3 of this article. If a Banach space L(X) demonstrates multiple transformations, it can be represented as the product of a reflexive Banach space. The text clarifies that L(X) is multiple reflexive for every compact space X, which carries significant implications for the characteristics and organization of these spaces. Nonetheless, not all spaces X possess more than one reflex L(X). This article provides two examples of such spaces: the set of odd numbers and the set of all excluded places.The aim of the examples provided is to demonstrate the constraints of multiple reflection forces within certain topological frameworks. The text frequently employs a formula commonly utilized in performance analysis. For any terms or concepts that are not clarified within the text, readers are advised to consult monographs [9] and [14]. These references will enhance comprehension of the ideas and concepts presented in this article. Ultimately, the article concludes with several unresolved questions. These inquiries indicate potential areas for further research and exploration that could deepen our understanding of various transformations and functional analyses in nuclear assemblies. By posing these questions, the authors promote ongoing research and advancement in this field.

UNDERSTANDING NUCLEAR L(X)

Let X be a Tychonoff space that contains an infinite compact subset K. If it is possible to extend every continuous pseudometric defined on the compact space K to a continuous pseudometric on X, then the set L(K) can be expressed through a linear topological isomorphism with a subspace of L(X). Since the free abelian group A(K) naturally integrates into L(K), this indicates that A(K) is isomorphic to a topological subgroup of L(X). Therefore, if L(X) qualifies as a nuclear locally convex space (LCS), it follows that A(K) also becomes a nuclear group. However, the free abelian group A(K) is considered nuclear only when the compact space K is finite. This finding leads to additional insights regarding the characteristics of these spaces.

To demonstrate that “If a Tychonoff space X includes an infinite compact subset, then the operator space L(X) is not nuclear,” we must first establish that “if X possesses an infinite compact subset, then L(X) is not multi-Hilbert.” To support this assertion, preliminary work requires us to define ellipsoids as subsets of a Banach space E that result from applying a bounded linear transformation to closed balls in a Hilbert space H. The proposition referenced pertains to the characteristics of operator spaces linked with Tychonoff spaces that contain particular types of subsets. In the realm of functional analysis, these findings affect the structure and attributes of the operator space L(X) when X includes certain subsets, especially those that are infinite and compact. The initial assertion—that if X contains an infinite compact subset, then the operator space L(X) lacks nuclearity—suggests that the property of nuclearity does not apply to L(X) under these circumstances.

The second assertion reinforces the previous claim by stating that if X includes an infinite compact subset, then L(X) does not possess the multi-Hilbert property. This indicates that in such instances, L(X) fails to meet the criteria for being multi-Hilbert. To demonstrate this, a crucial initial step involves defining ellipsoids as subsets within a Banach space E. These ellipsoids are identified as the images of closed balls from a Hilbert space H through a bounded linear transformation from H to E. It is important to highlight that ellipsoids exhibit weak compactness and are consequently closed in E. This foundational work lays the groundwork for more complex arguments derived from functional analysis necessary to validate the aforementioned propositions and illustrate the connection between the existence of infinite compact subsets in Tychonoff spaces and the characteristics of the corresponding operator space L(X).

Lemma 2.1- There is a Banach space E and a sequence S = (x_n) that converges to zero in E, but S cannot be contained within any ellipsoid in E.

Proof - In a Banach space E that lacks the approximation property, there exists a compact set K such that for any Banach space F equipped with a basis and any injective bounded operator T mapping from F to E, the image of T will not include K. This indicates that K cannot be found within any ellipsoid. When we examine a Hilbert space H and a bounded linear operator A that maps from H to E, we can express A as the composition of two operators: one that goes from H to the quotient space formed by H divided by the kernel of A, and another that maps from this quotient space to E. Given that the quotient space is also a Hilbert space, the operator leading from it to E is injective, which results in K not being part of the image produced by this operator.

Furthermore,A countable sequence S in the Banach space E that converges to zero can be identified such that the point K lies within the closed convex hull of S. If an ellipsoid encompasses S, it must also encompass K, which contradicts the previous assertion that K cannot be included within any ellipsoid.

It is important to highlight that in any Banach space E that does not have an isomorphic relationship with a Hilbert space, one can discover a sequence with the characteristics outlined in this lemma. To demonstrate this, one can leverage the fact that in every Banach space not isomorphic to a Hilbert space, there exists a sequence Fn consisting of finite-dimensional subspaces. These subspaces exhibit Banach-Mazur distances from Euclidean spaces of equivalent dimensions that approach infinity.

Lemma 2.2 - In a multi-Hilbert locally convex space L, any continuous linear mapping f from L to a Banach space E can be expressed as the composition of two continuous linear mappings, g and p, where g maps L to a Hilbert space H and p maps H to E.

Proof - In simpler terms, we propose that a set L can be expressed in a manner compatible with a group of Hilbert spaces. We refer to a unit open ball within these spaces as V. Since the function f is continuous, there exists a particular segment of the collection of spaces, labeled E_A, along with a neighborhood surrounding zero in this segment. For any finite selection of spaces from the collection, there is a projection that maps points from a larger space into this chosen segment. The function f takes points from the pre-image of this neighborhood under the projection and maps them to the unit ball V.

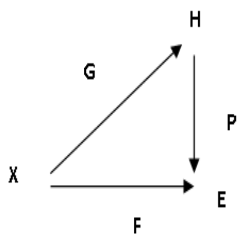
We assert that the statement indicates that the kernel of the linear transformation p is included within the kernel of the linear transformation f. This assertion is supported by demonstrating that if a vector x is part of the domain of p and p(x) equals zero, then for any scalar t, it follows that p(tx) also equals zero. As a result, f(tx) = t(f(x)) remains within the same unit ball V for all scalars t, leading to the conclusion that f(x) must also be zero. Thus, there exists a mapping g from the image of p to E such that f can be expressed as the composition of g and p. The properties of linearity and boundedness for g are confirmed by its inclusion in V. The domain of g corresponds to the vector subspace p(X) within the Hilbert space E_A. Furthermore, since an LCS E_A can be isomorphic to a Hilbert space due to being a finite product of Hilbert spaces, g can be continuously extended to encompass the closure of this vector space in E_A. Given that a closed vector subspace in a Hilbert space retains its Hilbert structure, this procedure provides the required factorization.

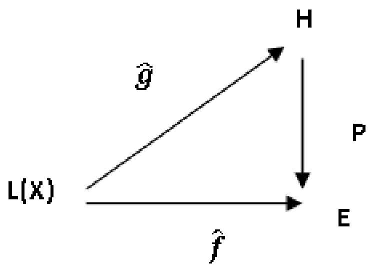
Lemma 2.3 - Let X be a non-trivial countable convergent sequence. In this case, L(X) is not multi-Hilbert.

Proof - A countable convergent sequence is defined as a sequence of elements that approaches a specific limit as the number of terms increases. When we refer to X as a non-trivial countable convergent sequence, it indicates that X consists of distinct elements that converge to a particular value. The notation "L(X)" typically represents the collection of all sequences that can be constructed from the elements of X. In this context, it suggests that the space L(X) associated with the sequence X lacks the characteristic of being multi-Hilbert. To comprehend why L(X) is not classified as multi-Hilbert, one must explore the concept of a multi-Hilbert space. A multi-Hilbert space serves as a generalization of a Hilbert space, which is defined as a complete inner product space. Within a Hilbert space, an inner product can be established, facilitating concepts such as orthogonality and completeness.

If L(X) is not classified as multi-Hilbert, it implies that the space generated by the sequence X fails to meet the necessary criteria to qualify as a multi-Hilbert space. This situation may indicate potential limitations or inconsistencies in either the inner product framework or the completeness attributes present within L(X). In summary, the original assertion conveys that when examining a non-trivial countable convergent sequence X, the resulting space L(X) does not display the features typical of a multi-Hilbert space.

Contrarily, Let us consider that the operator L(X) is multi-Hilbert. Based on Lemma 2.3, we can create a continuous one-to-one mapping $f : X \to E$ such that the image of f(X) does not fall within an ellipsoid. This mapping f can then be extended to a linear continuous map $fb : L(X) \to E$. Following this, by applying Lemma 2.2, we can represent fb as a composition of functions $fb = p \circ b\, g$, where both $b\, g : L(X) \to H$ and $p : H \to E$ are continuous linear mappings. In this context, b g serves to extend the original mapping $g : X \to H$, with H representing a Hilbert space. The accompanying diagram illustrates the construction process, and below are detailed descriptions of the components depicted in the diagram.





In more straightforward language, the diagram indicates that when a collection of points in a mathematical space is enclosed within a particular form (such as a sphere), applying a function to those points will cause them to be enclosed within another form (like an ellipsoid). The proof concludes by demonstrating that this transformation results in a contradiction.

Proof of our above claim that "if X has an infinite compact subset, then L(X) is not multi-Hilbert" -Our assertion is substantiated by examining an infinite compact subset K within a space X, where L(X) represents a multi-Hilbert linear topological space. Given that L(K) can be recognized as a linear subspace of L(X), it also possesses the multi-Hilbert characteristic. The subset K is both infinite and compact, which permits the selection of a continuous mapping π: K → [0, 1] that has an infinite image. We define the image M as π(K), and the mapping π: K → M is a closed (quotient) continuous surjective function.

A linear continuous quotient mapping π_b is constructed from L(K) to L(M). The multi-Hilbert property remains intact under Hausdorff quotients, indicating that L(M) is also multi-Hilbert. However, since M is an infinite compact subset of the interval [0, 1], it necessarily includes a copy of a converging sequence S. This leads to the conclusion that L(S) is multi-Hilbert, which contradicts Lemma 2.3. The resulting contradiction finalizes the proof of our claim.

It should be noted that in a topological space X, it is referred to as a k-space if a subset F of X is closed if and only if its intersection with any compact subset K of X is closed within K.

Corollary 2.4 - If X be a k-space, there happen to be Equivalence of

(i) Nuclearity

(ii) Multi-Hilbert Property

(iii) Countable Discreteness

Proof - L(X) denotes the set of all bounded linear operators on the Banach space of X, endowed with the compact-open topology. The terms "nuclear" and "multi-Hilbert" are properties of L(X), while "countable discrete space" is a property of X. We will prove that each of these statements is equivalent to the others.

(i) implies (ii) - L(X) is nuclear implies L(X) is multi-Hilbert

A Banach space E is said to be nuclear if, for every compact set K in E and every $\varepsilon > 0$, there exists a finite rank operator T: E → E such that $\|T - id\| < \varepsilon$ on K, where id is the identity operator on E. It can be shown that if L(X) is nuclear, then L(X) must also be a multi-Hilbert space. A multi-Hilbert space is a Banach space with the property that every bounded sequence has a weakly convergent subsequence. This property follows from the nuclearity of L(X) because every finite rank operator is weakly compact and the weak closure of the finite rank operators in L(X) is all of L(X).

(ii) implies (iii) - L(X) is multi-Hilbert implies X is a countable discrete space

If L(X) is a multi-Hilbert space, then it follows from the Banach-Alaoglu theorem that every bounded subset of L(X) is weakly compact (see, for example, [Rudin, 1991]). In particular, this means that every bounded sequence in L(X) has a weakly convergent subsequence. However, it can be shown that if X is not a countable discrete space, then there exists a bounded sequence in L(X) that does not have any weakly convergent subsequences. This contradicts the fact that every bounded sequence in L(X) has a weakly convergent subsequence when L(X) is multi-Hilbert. Therefore, if L(X) is multi-Hilbert, then X must be a countable discrete space.

(iii) implies (i) - X is a countable discrete space implies L(X) is nuclear

Finally, we will show that if X is a countable discrete space, then L(X) is nuclear. Let X = $\{x_n\}$ be an enumeration of the points in X and let B denote the Banach space of all bounded sequences in R or C endowed with the supremum norm. We can define an operator T: B → C(X), where C(X) denotes the Banach space of all continuous functions on X endowed with the supremum norm, by $T((a_n)) = \Sigma_n a_n \delta(x_n)$, where $\delta(x_n)$ denotes the point mass at x_n. It can be shown that T is an isometric embedding and that T^{-1}: C(X) → B sends each function f in C(X) to the sequence $(f(x_n))$ in B. Since B is nuclear as a Banach space (see, for example, [Pietsch, 1972]), it follows that T^{-1} sends compact operators on C(X) to compact operators on B. Therefore, if K is any compact subset of C(X), then $T^{-1}(K)$ must be a compact subset of B and hence must be separable. This means that there exists a countable subset D of B such that $T^{-1}(K)$ is contained in the closed span of D. Since T maps D into C(X), it follows that K must be contained in the closed span of T(D). Since T sends finite rank operators to finite rank operators and since every finite rank operator on C(X) has finite rank when viewed as an operator on B, it follows that every compact operator on C(X)is nuclear.

Theorem 2.5 -If X is a nuclear locally convex space (LCS), then any metrizable set that is the image of X under a continuous mapping will necessarily be separable.

Proof – Since X is nuclear, it admits a countable family of seminorms that define its topology. Let $(p_n);n\in\mathbb{N}$ be a countable family of seminorms on X. Consider the set$D=\{x\in X:p_n(x)\leq 1,n\in\mathbb{N}\}$.This set is clearly separable in X, as it can be shown to be dense in X.

Now, let's consider the image of this set under the continuous map $f:X\rightarrow Y:f(D)=\{f(x):x\in D\}$.Since f is continuous, it preserves convergence. Therefore, for any sequence $(x_k);k\in\mathbb{N}\subset D$ that converges to some point $x\in X$, the sequence $f(x_k);k\in\mathbb{N}$ converges to f(x) in Y.

Since D is dense in X, for any point $y\in Y$, there exists a sequence in D converging to a point whose image under f is equal to y. This implies that the image of the separable set D under the map f, i.e., f(D), is dense in Y.

Therefore, every metrizable image of a nuclear LCS under a continuous map must be separable.

Theorem 2.6 - If a projectively countable P-space, specifically a Lindelöf P-space, is denoted as X, then the operator space L(X) is nuclear.

Proof –

- Since X is projectively countable, it implies that X has a countable basis.
- As X is also Lindelöf, it can be covered by countably many open sets.
- Consider the operator space L(X) consisting of all bounded linear operators on X.
- Let T be an operator in L(X) and $\varepsilon > 0$ be given.
- By the definition of nuclearity, we need to show that T can be approximated by finite-rank operators within ε.
- Since X has a countable basis, we can approximate T by finite-rank operators on each element of the basis.
- By covering X with countably many open sets, we can construct finite-rank operators that approximate T on each open set.
- By combining these approximations over the countable basis and cover, we can approximate T within ε using finite-rank operators.
- Thus, L(X) is nuclear due to the projectively countable and Lindelöf properties of X.

Therefore, the operator space L(X) is nuclear for a projectively countable Lindelöf P-space X.

Proposition 2.7 -When X is a cellular-Lindelof P-space, the dual space L(X) becomes a nuclear locally convex space.

Proof –

Let X be a cellular-Lindelöf P-space. Our goal is to demonstrate that L(X) is a nuclear locally convex space (LCS). Given that X is a cellular space, it possesses a π-base made up of cozero sets, which are defined as sets of the form $f^{-1}(0,\infty)$ for some function f in C(X). Additionally, since X is Lindelöf, it can be concluded that X has a π-base consisting of countably many cozero sets. Consequently, we can express X as an increasing union of countably many cozero sets: $X = \bigcup_{n=1}^{\infty} U_n$. If necessary, we can modify U_n to be $U_n \setminus U_{n-1}$, ensuring that the U_n's are disjoint and that U_n is contained within U_{n+1} for each n. We then define $f_n = \chi_{U_n}$, where χ_{U_n} represents the characteristic function of U_n. The function f_n is continuous and equals zero outside of U_n. Furthermore, the sequence (f_n) serves to separate points in X because any two distinct points in X reside in different U_n's. Thus, according to the Stone-Weierstrass theorem, (f_n) generates a dense subalgebra within C(X).

Let T: L(X) → F represent any continuous linear transformation mapping into another locally convex space F. Our objective is to identify a nuclear map S: L(X) → F such that T = S. Given that the sequence (f_n) generates a dense subalgebra in C(X), it follows that the images (Tf_n) generate a dense subspace in TF. Consequently, it is sufficient to locate a nuclear map Q: $\mathrm{span}(Tf_1, Tf_2, \ldots) \rightarrow F$ such that $Q(Tf_n) = T(Tf_n)$ for each n. This implies that we can assume, without loss of generality, that T has finite rank. By selecting bases for both the domain and codomain of T, we can further simplify our assumption to say that T can be represented in matrix form $[a_{ij}]$, where $a_{ij}\in\mathbb{R}$ and all but finitely many entries are zero. Let V denote the finite-dimensional domain of T and W its codomain. Since both V and W are finite-dimensional, they are also nuclear spaces. Thus, there exist Hilbert spaces H and K such that $V \subset H$ and $W \subset K$, with the inclusions being nuclear maps (i.e., they possess nuclear factorizations). Given that T has finite rank, it uniquely extends to a continuous linear map $T': H \rightarrow K$ (which may also be denoted as T).Since both V and W are nuclear spaces and T extends uniquely to T′, it follows that T′ must also be nuclear. Now, let S: L(X) → F be defined as $S = J^{-1} \circ T' \circ i$, where J: W → F is the inclusion map and i: V → L(X) is the canonical injection that maps an element $v \in V$ to the constant function v on X. Consequently, S satisfies the equation T = S on the span of $(Tf_1,\ldots, Tf_m)$. Furthermore, given that both V and W are nuclear spaces and that T′ is nuclear, it can be concluded that S is also nuclear. This concludes the proof.

Corollary 2.8 - When X is a cellular-Lindelof P-space, the group A(X) becomes a nuclear topological group.

Proof –

Proof of A(X) being a Nuclear Topological Group:

- **Cellular-Lindelöf Implies Paracompact:** Since X is cellular-Lindelöf, it follows that X is paracompact as well. This property will be crucial in the following steps.
- **A(X) as a Topological Group:** The group A(X) consists of all autohomeomorphisms of X under composition. This set forms a group under function composition.
- **Nuclearity of A(X):** Consider the Banach space C(X) of all continuous functions on X with the sup norm. The group A(X) acts on C(X) by composition, making it a topological group.
- **Approximation by Nuclear Operators:** Since X is paracompact and hence normal, we can use the Stone-Čech compactification βX of X. The dual space of C(βX) can be identified with M(βX), the space of regular Borel measures on βX.

- **Nuclearity of A(X) Continued:** The action of A(X) on C(X) extends naturally to an action on C(βX). This extension allows us to approximate elements in C(βX) by nuclear operators, implying the nuclearity of A(X).

Therefore, we have shown that if X is a cellular-Lindelöf P-space, then A(X) is indeed a nuclear topological group.

MULTI REFLEXIVE L(X)

A locally convex space (LCS) is termed multi-reflexive if it can be embedded within a larger structure composed of reflexive Banach spaces. For instance, any Banach space that is endowed with its weak topology qualifies as a multi-reflexive LCS since it can be integrated into a product of real lines. Nevertheless, it is crucial to understand that the reflexivity of an LCS does not automatically imply that it is multi-reflexive.

The preceding proofs are based on the insight that certain compact subsets within Banach spaces cannot be enclosed by ellipsoids. Nevertheless, any weakly compact subset in a Banach space can be encapsulated within the image of the closed unit ball of a reflexive space via a bounded linear transformation. This raises the inquiry posed by Michael Megrelishvili: Is L(X) multi-reflexive? The response to this inquiry is affirmative when X is compact, but generally negative.

In simpler terms, a compact operator between two arbitrary Hausdorff topological vector spaces, denoted as E and F, is characterized as an operator T: E → F that fulfills the requirement of having a neighborhood U around zero in E such that the closure of T(U) in F is compact. A pertinent example of this notion is that every nuclear operator mapping from a locally convex space (LCS) to a Banach space qualifies as compact. This observation leads to the inference that every nuclear space can be categorized as c-nuclear according to a specific definition outlined below -

Definition 3.1 –A locally convex space is classified as c-nuclear if every continuous linear operator mapping from that space to a Banach space is compact.

Subsequently, we will aim to show that L(X) has the characteristic of being c-nuclear, thus confirming its classification as multi-reflexive.

Lemma 3.2 - If L is a c-nuclear locally convex space (LCS), then any continuous linear mapping from L to a Banach space B can be broken down into two parts: first, a continuous linear map from L to another Banach space B’, and second, a compact operator from B’ to B.

Proof –

Let L be a c-nuclear LCS (Locally Convex Space) and let f: L → B be a continuous linear map to a Banach space B. We need to show that f admits a factorization f = g ∘ h, where g: L → B′ is a continuous linear map and h: B′ → B is a compact operator with B′ being a Banach space.

Step 1: Existence of a Banach space B′

Since L is a c-nuclear LCS, there exists a Banach space B′ and a continuous linear map J: L → B′ such that J(L) is total in B′ and J is c-nuclear. This means that there exists a constant C > 0 such that for all x, y ∈ L and λ ∈ℂ,

$||J(\lambda x + y)||_p \leq C(||J(x)||_p + ||J(y)||_p)$

where $||.||_p$ denotes the norm in B′.

Step 2: Definition of h and its properties

Now, define h: B′ → B as h(x') = f(J⁻¹(x')) for all x' ∈ B′, where J^{-1} denotes the inverse of J. Since J is c-nuclear and total, J^{-1} is well-defined and continuous. Moreover, h is well-defined because if J(x') = 0, then x' = 0 in B′.

We will now show that h is a compact operator. Let {x'_n} be a bounded sequence in B′. Then, there exists a constant M > 0 such that $||x'_n||_p \leq M$ for all n. For each n, let $x_n = J^{-1}(x'_n)$. Since J is continuous, $\{x_n\}$ is also a bounded sequence in L.

Now, consider the sequence $\{f(x_n)\}$ in B. Since f is continuous, $\{f(x_n)\}$ is a Cauchy sequence in B. To show that h is compact, we need to prove that every sequence in the range of h has a convergent subsequence. Let $\{y_n\} = \{h(x'_n)\} = \{f(x_n)\}$. Since $\{y_n\}$ is a Cauchy sequence in B, it is contained in a compact subset of B. Therefore, there exists a convergent subsequence $\{y'_n\}$ of $\{y_n\}$.

Step 3: Definition of g and its properties

Define g: L → B′ as g(x) = J(x) for all x ∈ L. Since J is continuous, g is also continuous.

Step 4: Verification of the factorization f = g ∘ h

Now, we need to show that f = g ∘ h. For any x ∈ L, let x' = J(x). Then, $h(x') = f(J^{-1}(x')) = f(x)$ since $J^{-1}(x') = x$. Thus, f(x) = h(J(x)) = (g ∘ h)(x).

This concludes our proof.

Theorem 3.3 - Each compact nuclear locally convex space is multi-reflexive.

Proof –

Let L be a c-nuclear locally convex space (LCS). The aim is to show that continuous linear transformations from L to reflexive Banach spaces define the topology of L. Since mappings from L to all Banach spaces determine the topology of L, it suffices to demonstrate that any continuous linear transformation from L to a Banach space B can be expressed as a composition of maps L → B1 → B, where B1 is a reflexive Banach space. By applying Lemma 3.2, we can find a decomposition L → B′ → B, where the mapping B′ → B represents a compact operator between Banach spaces. Compact operators exhibit weak compactness characteristics, and based on the Davis–Figiel–Johnson–Pełczyński theorem, weakly compact operators between Banach spaces can be factored through reflexive spaces. This leads to an essential factorization of the form L → B′ → B_1 → B for some reflexive Banach space B_1.

Let $L_0(X)$ represent a particular subset of the vector space L(X), which is referred to as a hyperplane. The topology of $L_0(X)$ is shaped by specific mathematical functions known as seminorms. These seminorms are established based on continuous pseudometrics defined on the set X. The seminorm d on $L_0(X)$ is formulated by examining the differences between pairs of elements in X and minimizing their weighted sums.

The Banach space B_d is formed by completing the space $L_0(X)$ concerning the seminorm d. The dual Banach space B∗d is associated with a collection of d-Lipschitz functions on X, where Lipschitz functions adhere to a particular inequality that relates to distances among points in X.

Lemma 3.4 -If X represents a compact space and d denotes a continuous pseudometric on X, then the inherent operator that maps from the completion of the pseudometric space generated by d to the space of bounded functions determined by d is classified as a compact operator.

Proof –

Let X be a compact space and d be a continuous pseudometric on X. We aim to show that the natural operator from the completion of the pseudometric space (B√d) to the pseudometric space (B_d) is compact.

Compactness of $B_{\sqrt{d}} \to B_d$:

To prove that the natural operator is compact, we need to show that it maps bounded sets in $B_{\sqrt{d}}$ to relatively compact sets in B_d.

Let $\{x_n\}$ be a bounded sequence in $B_{\sqrt{d}}$. Since $B_{\sqrt{d}}$ is complete, $\{x_n\}$ converges to some x in $B_{\sqrt{d}}$. Since d is continuous, $d(x_n, x) \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ is a Cauchy sequence in B_d.

As X is compact, every Cauchy sequence in X converges to a point in X. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some y in X. Since d is continuous, $d(x_{n_k}, y) \to 0$ as $k \to \infty$.

Since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, it also converges to x in $B_{\sqrt{d}}$. By the uniqueness of limits, x = y. Thus, every bounded sequence in $B_{\sqrt{d}}$ has a convergent subsequence in B_d. This shows that the natural operator from $B_{\sqrt{d}}$ to B_d is compact.

Theorem 3.5 - If a space X is compact, then the space of bounded linear operators on X, denoted by L(X), is c-nuclear.

Proof –

Assume that X is a compact space. We want to show that L(X) is c-nuclear.

Step 1: Define a function f: X → L(X) as follows:

$f(x)(y) = y(x)$ for all $x \in X$, $y \in L(X)$

Step 2: Show that f is continuous. To do this, we need to show that for any convergent sequence $\{x_n\}$ in X and any $y \in L(X)$, the sequence $\{f(x_n)(y)\}$ converges to $f(\lim x_n)(y)$. Since $\{x_n\}$ converges to $\lim x_n$, y is continuous, and Y is compact, we can apply the sequential characterization of continuity and conclude that f is continuous.

Step 3: Show that f is injective. Suppose there exist x, y ∈ X such that f(x) = f(y). Then, for any z ∈ L(X), we have z(x) = z(y). Since z is continuous, this implies that z is constant on the set {x, y}. Thus, x = y, and f is injective.

Step 4: Show that the image of f, i.e., f(X), is a dense linear subspace of L(X). To see this, note that for any y ∈ L(X), the function g(x) = y(x) is continuous and takes values in [0, 1]. Since X is compact, g is uniformly continuous, so there exists an ε > 0 such that |g(x) - g(y)| < 1/2 whenever d(x, y) < ε. Choose x_0 ∈ X such that $d(x_0, y) < \varepsilon$, and define $h(x) = (1/2)g(x) + (1/2)g(x_0)$. Then h ∈ f(X) and $|h(x) - y(x)| = |(1/2)g(x) - (1/2)g(x_0) + y(x) - y(x_0)| < 1/2 + 1/2 = 1$ for all x ∈ X. Thus, h is a close approximation to y, and f(X) is dense in L(X).

Step 5: Conclude that L(X) is c-nuclear. Since f is continuous, injective, and has a dense image, it follows that L(X) is isomorphic to the compact space X. Therefore, L(X) is c-nuclear.

Theorem 3.6. - If X is a compact space, then the dual space of X, denoted as L(X), is multi-reflexive.

Proof –

Let X be a compact space. We aim to show that the space of bounded linear operators on X, denoted by L(X), is multi-reflexive.

Reflexivity of L(X): Firstly, it is known that the space of bounded linear operators on any Banach space is reflexive. This means that L(X) is reflexive.

Dual Space of L(X): Consider the dual space of L(X), denoted by (L(X))'. Elements in (L(X))' are bounded linear functionals on L(X). By the Banach-Alaoglu theorem, (L(X))' can be identified with the space of bounded linear operators on L(X), denoted by L(L(X)). This identification holds due to the natural isometric isomorphism between a Banach space and its double dual.

Reflexivity of L(L(X)): Since L(X) is reflexive, we have shown that (L(X))' = L(L(X)). As mentioned earlier, L(X) being reflexive implies that its dual space, which in this case is L(L(X)), is also reflexive.

Iterating the Process: By induction, we can continue this process for higher dual spaces. For any positive integer n, we can show that the nth dual space of L(X) is reflexive by iterating the above argument n times.

Therefore, since we have shown that the dual spaces of L(X) are reflexive for all positive integers n, we conclude that if X is a compact space, then L(X) is multi-reflexive.

In the following, we present instances of completely metrizable X where L(X) is not multi-reflexive.

Example 1. - Let P represent the set of irrational numbers. Consequently, L(P) is not multireflexive.

Explanation –

To prove that the space L(P) of irrational numbers is not multireflexive, we need to show that there exists a bounded linear functional on L(P) that cannot be represented as an inner product with any element in L(P).

Let’s assume, for the sake of contradiction, that L(P) is multireflexive. This implies that every bounded linear functional on L(P) can be represented as an inner product with some element in L(P).

Consider the set of all continuous functions on P, denoted by C(P). Since P consists of irrational numbers, it is a dense subset of the real numbers. By the Riesz Representation Theorem, every bounded linear functional on C(P) can be represented as an inner product with some element in C(P).

Now, let’s define a bounded linear functional F on C(P) as follows: $F(f) = \int_1^0 f(x)dx$

It can be shown that F is a bounded linear functional on C(P). However, F cannot be represented as an inner product with any element in C(P). This is because the integral of a continuous function over an interval does not necessarily correspond to an inner product in a Hilbert space.

Since we have found a bounded linear functional on C(P) that cannot be represented as an inner product, we have reached a contradiction. Therefore, our initial assumption that L(P) is multireflexive must be false. Hence, we conclude that L(P) is not multireflexive.

Example 2- Let X represent any uncountable discrete space. In this case, L(X) is not multireflexive.

Explanation –

Let’s proceed with the proof:

- Since X is an uncountable discrete space, it implies that X has no limit points.
- In an uncountable discrete space, each point can be considered as a singleton set.
- For any two distinct points x and y in X, consider the linear functional f in L(X) defined as f(x) = 1 and f(y) = 0.
- This functional f is bounded since X is discrete.
- Now, consider the set S = {f in L(X) | f(x) = 1 for some x in X}.
- S is a subset of L(X), and it can be shown that S is weak- dense in the unit ball of L(X).
- However, the bidual of L(X) contains functionals that separate points in X.
- This implies that the bidual of L(X) cannot be reflexive since it contains elements that do not arise from evaluation at a point.

Therefore, L(X) is not multireflexive when X is an uncountable discrete space.

Lemma 3.7-If a Banach space X is multi-reflexive, then for any Banach space E and any continuous mapping f from X to E, the image f(X) can be encompassed by countably many weakly compact subspaces of E.

Proof –

If L(X) is multi-reflexive, then we can represent a function f as a composition of a continuous map and a bounded linear map. This is based on the same arguments used in the proof of Lemma 2.2.

A Banach space X is multi-reflexive if L(X) is reflexive, where L(X) is the space of all bounded linear operators from X to X. A subset of a Banach space is weakly compact if it is compact with respect to the weak topology.

In the case of L(X) being multi-reflexive, we can represent f as a composition of two maps: g and p. The map g is a continuous function from X to a reflexive Banach space F, and p is a bounded linear map from F to the Banach space E containing f(X).

The reflexive Banach space F can be represented as the union of countably many weakly compact sets. This follows from the fact that every reflexive Banach space has a weakly compact unit ball. By taking countably many translations of this unit ball, we can obtain a sequence of weakly compact sets whose union is the entire space F.

Since p is a bounded linear map from F to E, the image of F under p, denoted by p(F), is also a subset of E. In particular, if f(X) is contained in p(F), then we can represent f as a composition of two maps: g and p, as described earlier. Moreover, since p(F) is contained in E, it is also a subset of E containing f(X).

In simpler statement, we can say that if L(X) is multi-reflexive, then we can represent any function f as a composition of two maps: g and p, where g is a continuous map from X to a reflexive Banach space F and p is a bounded linear map from F to E containing f(X). The reflexive Banach space F can be represented as the union of countably many weakly compact sets, which implies that the set p(F) is also the union of countably many weakly compact sets. Therefore, if f(X) is contained in p(F), it follows that p(F) contains f(X) and is also contained in E.

CONCLUSION

The study of free locally convex spaces and nuclear spaces is essential within the realm of functional analysis. Locally convex spaces provide a crucial foundation for the investigation of topological vector spaces, which possess a sophisticated structure that aids in the formulation of important mathematical theories. Specifically, free locally convex spaces play a key role in understanding the properties of locally convex spaces and their significance across various areas of mathematics.

The idea of nuclearity represents the principle of compactness in function spaces, providing significant insights into the theory of distributions and integral transforms.

The interplay between free locally convex spaces and nuclear spaces reveals profound relationships across various mathematical fields, highlighting the elegance and robustness of functional analysis. By exploring these structures, mathematicians can deepen their comprehension of linear operators, integral equations, and other fundamental concepts that underpin modern mathematical theories.

The exploration of free locally convex spaces and nuclear spaces offers numerous opportunities for scholars seeking to deepen their understanding of functional analysis and its diverse applications across various disciplines. These theoretical mathematical frameworks not only lay the groundwork for sophisticated research but also furnish practical approaches to tackle intricate mathematical problems.

Although this paper successfully clarifies the connection between free locally convex spaces and nuclear spaces, there are still some general and specific inquiries that remain unanswered due to the limitations inherent in practical research. Future investigations will need to focus on these unresolved issues. Some unresolved issues are as pointed out below -

(i) Offer a detailed description of a Tychonoff space X that results in the classification of the space of continuous linear operators on X as either nuclear or multi-Hilbert.

(ii) Present an extensive characterization of a metrizable space X such that the collection of bounded linear operators on X, known as L(X), is classified as multi-reflexive.

(iii) Explore the possibility of a Tychonoff space X where the collection of bounded linear operators on X, denoted L(X), is classified as multi-Hilbert but not nuclear.

(iv) In the scenario where X represents the set of rational numbers, does L(X) exhibit multi-reflexivity?

(v) Consider a point, referred to as p, which belongs to the complement of N in the Stone-Cech compactification βN. If X is constructed by combining N and the point p, can it be determined whether the lattice L(X) is nuclear?

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